Primordial Non-Gaussianity and the EFT of Inflation

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If the universe were simply the motion which follows from a given scheme of equations of motion with trivial initial conditions, it could not contain the complexity we observe. Quantum mechanics provides an escape from the difficulty. It enables us to ascribe the complexity to the quantum jumps, lying outside the scheme of equations of motion. The quantum jumps now form the uncalculable part of natural phenomena, to replace the initial conditions of the old mechanistic view.

—Paul A.M. Dirac, 1939

Acknowledgements

I would like to acknowledge and thank my supervisor João Magueijo for attending to my questions and concerns this summer, and especially for allowing me the license to indulge in topics of the field which I found interesting at the time. This process would not have been as rewarding otherwise. I am also very grateful to Carlo Contaldi and Jonathan Halliwell for taking the time to answer a few of my questions during this process, and I owe much gratitude to Daniel Baumann for the figures used in this paper.

In little over two months, what was planned to be a small treatise on a rather concise topic quickly exploded into the overview today ("inflationary expansion" perhaps?), which delves into aspects diverse as problems with constructing consistent inflationary models to examining cosmic 21-cm radiation. Rather than simply fulfilling a requirement for the MSc degree, the process of rummaging through arXiv articles and composing this dissertation has strongly motivated what I would like to work on. Hopefully some readers will find this paper as constructive as I found writing it.

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(Figures where taken from notes concerning inflation *TASI Lectures on Inflation* by Daniel Baumann [12] for illustrative purposes: Figure 17 on p. 69 and Figure 30 on p. 94. Tables were taken from [207]: Table 1 from p. 10 and Table 2 from p. 28.)

Nomenclature

The metric signature is $(-,+,+,+)$. We use natural units $c = \hbar = 1$ and the reduced Planck mass $M_{\rm Pl} \equiv (8\pi G)^{-1/2}$ is normally set to 1.

Chapter 1

Introduction

Spoken during the relative adolescence of quantum theory, Dirac's remarkable insight stands today as almost prescient of modern cosmological understanding: the initial conditions of our universe, which evolve into the staggering complexity of clusters and superclusters of galaxies observed today, necessitate the existence of quantum mechanics. Not only is this recognition simply extraordinary for its perspicacity, but we should note that it was also spoken during the infancy of scientific cosmology, only a few years after Hubble's discovery of the expansion of the universe in 1929, and the presentation of the Friedman-Lemaître-Robertson-Walker (FLRW) metric modeling a homogeneous and isotropic universe. Nearly a half-century later, cosmologists subsequently accepted that the initial seeds of structure were sourced by quantum fluctuations, and have, by and large, gradually coalesced around a standard paradigm for this period in the very early universe: inflation.

The inflationary epoch is a hypothetical period of exponential expansion by a factor of 10^{26} which takes place approximately 10^{-34} seconds after the Big Bang singularity, driving the universe towards a homogeneous and spatially flat FLRW cosmology. We should note that this in itself is arguably an impressive achievement, as when inflation was first introduced, the flatness of the universe was not conclusively determined by observational data. Furthermore, the quantum fluctuations during this period result in inhomogeneities, both the initial seeds of large-scale structure and temperature anisotropies of the cosmic microwave background.

The past two decades have been celebrated as the *Golden Age* of precision cosmology, as observational cosmology has measured and constrained the parameters of our universe to a precision never before conceivable. Much of this knowledge results from experiments probing the temperature fluctuations of the cosmic microwave background

(CMB). This is the remnant radiation of the last-scattering surface from when electrons and protons combined to form neutral hydrogen and free-traveling photons (i.e. the era of *recombination*), approximately 379,000 years after the Big Bang. Although measured to be extremely isotropic, measuring the small fluctuations of the CMB temperature map (of the order $\Delta/T \sim 10^{-5}$) allows us to both characterize the observed universe, but also investigate the conditions before the CMB formed, the period after the Big Bang. Namely, these inhomogeneities are seen as the result of inflation, and remain the most promising window to understanding this little-known period in time.

Such measurements from the CMB and surveys of large-scale structure (LSS) have resulted in the concordance model of cosmology, remarkably parametrized by only a handful of variables, detailed in the table below [15,207]:

Parameter	**Physical Origin**	**Value**
Baryon Fraction Ω_b	Baryogenesis	0.0456 ± 0.0015
Dark Matter Fraction $\Omega_{\rm CDM}$	Physics at TeV-Scale?	0.228 ± 0.013
Cosmological Constant Ω_Λ	Unknown	0.726 ± 0.015
Optical Depth τ	First Stars	0.084 ± 0.016
Hubble Parameter h	Cosmological Epoch	0.705 ± 0.013
Scalar Amplitude A_s	Inflation	$(2.445 \pm 0.096) \times 10^{-9}$
Scalar Index n_s	Inflation	0.960 ± 0.013

Table 1.1: The parameters of the ΛCMD universe. Such a universe is spatially flat with the dark energy density of the form $\Omega_\Lambda = 1 - \Omega_b - \Omega_{\rm CDM}$. Note that the Hubble parameter h parametrizes the expansion rate of the present universe, such that $H_0 = 100h$ km s^{-1}Mpc^{-1}. The inflationary parameters above concern the primordial power spectrum of scalar perturbations; we will discuss this in full below.

Cosmologists today are able to confidently specify that we live in a flat ΛCMD universe, which began as a hot dense state as predicted by the Big Bang, composed of a homogeneous background of 4.6% baryonic matter, 22.8% dark matter, and 74.2% so-called "dark energy". By observing the systematically redshifted light from distant galaxies and supernovae, we can confirm that the universe is manifestly expanding, and the abundances of light elements H, He, D, and Li mirror predications of Big Bang Nucleosynthesis.

However, how confident can we be in the inflationary paradigm? We still know practically nothing abut the specific mechanism by which inflation is driven. (For instance, what does an inflationary action look like precisely?) Moreover, over three decades of theoretical work has resulted in a myriad of inflationary models. Will it ever be possible to falsify some of these models in favor of others? Will we some day be able to assert that the inflationary epoch is a component of our standard cosmological model with the same confidence as, say, Big Bang Nucleosynthesis?

So far, the general inflationary scenario has passed a number of generic predictions as cosmological data has become increasingly precise. Namely, not only has the universe been found to be *flat* over cosmological scales, but primordial density fluctuations are observed to be *small, scale-invariant, Gaussian, adiabatic*, and *superhorizon*.[1]

Additionally, it should be observed that different inflationary models could predict different levels of inhomogeneity in the universe (via quantum fluctuations): the measurement of these departures from homogeneity could allow us to constrain inflationary models. Thus far, we have measured that the initial conditions of these density fluctuations are consistent with being a Gaussian random field. Future probes of the CMB (namely the Planck mission) are projected to survey the CMB with unprecedented precision: could we possibly find deviations from Gaussian statistics? If so, what kind of non-Gaussianity do we expect to find? Could we correlate such departures from Gaussianity with particular inflationary models, thereby "testing" them?

The first part of paper details the efforts to constrain inflationary models with a variety of means, particularly by non-Gaussianity. We detail the current and future observations which may be utilized to parametrize and possibly constrain the various inflationary models available. In addition, we mention how observations could possibly prefer the inflationary paradigm over other alternatives (e.g. with a significant detection of gravitational wave background), or possibly falsify the inflationary paradigm altogether.

The second part of this paper paper details a new approach to generalize inflationary models and classify them based upon their observational predictions. This is the "Effective Field Theory of Inflation", inspired by effective field theory methods used in high-energy particle physics. We present this method to write all single-field inflationary models in full generality, which also allows us to systematically explore non-Gaussian signatures of the inflationary action.

[1] Note that we will define and discuss these terms in full within the paper.

Chapter 2 concerns the dynamics of inflationary models and connections to observations. We begin by detailing slow-roll inflation, and then introduce the plethora of inflationary models and their behavior, expanding upon how observations may be able to parametrize and even discriminate between such models.

Chapter 3 details primordial non-Gaussianity, deviations from near Gaussian perturbations predicted by our simplest inflationary models. We will review how shapes of non-Gaussianity hold the promise of explicating the inflationary dynamics and could possibly falsify inflationary models.

Chapter 4 introduces the results of the Effective Field Theory of Inflation, and illustrates its advantages over other methods of inflationary model-building for exploring non-Gaussian signatures of inflationary dynamics.

Chapter 5 concludes with future observational to search for non-Gaussian signals and future theoretical work to improve techniques for constraining inflation.

(Note that we will largely ignore issues concerning large scale structure LSS in this paper for the sake of brevity).

Chapter 2

Inflationary Models and Observations

(Motivation and background for inflation and is found in Appendix A, as well as the rudimentary basics of cosmology and CMB physics. The necessary results from cosmological perturbation theory are detailed in Appendix B. It is assumed in the following section that the reader is familiar with this.)

2.1 Inflationary Dynamics

Inflation [1-6] is a very unfamiliar physical phenomenon: within a fraction of a second the universe grew exponentially at an accelerating rate. In its simplest conception, we can introduce a potential energy $V(\phi)$ of a scalar field $\phi(t, \mathbf{x})$ (i.e. the "inflaton") which describes the change in energy density during inflation.

Recall that the three equivalent conditions for inflation are given by

$$\frac{d(aH)^{-1}}{dt} < 0 \implies \frac{d^2a}{dt^2} > 0 \implies p < -\frac{\rho}{3} \tag{2.1}$$

Inflationary acceleration $\ddot{a} > 0$ requires a negative pressure p and energy density ρ which slowly dilute, along with some exit mechanism into our standard Big Bang FLRW cosmology.

Acceleration occurs when the potential energy of the field $V(\phi)$ dominates over its kinetic energy, $\frac{1}{2}\dot{\phi}^2$, and thus inflation is driven by the vacuum energy of the inflaton field. Inflation ends at ϕ_{end} when the kinetic energy has grown to become comparable to the potential energy, $\frac{1}{2}\dot{\phi}^2 \approx V$. CMB fluctuations are created by quantum fluctuations

$\delta\phi$ over about 60 $e-$folds before the end of inflation. (We will justify this below). After these 60 $e-$folds of expansion, the inflationary process must somehow end; this is referred to as "reheating". At reheating, the scalar field oscillates around the minimum of the potential such that oscillations of the scalar field act like pressureless matter. The energy density of the inflaton and ϕ-particles then decay into radiation, thus resulting in the hot Big Bang. (Note that the theory of reheating and how inflation ends is fairly intricate, so we must largely ignore it in this paper. Readers are advised to see [7] for further details.)

The dynamics of a scalar field minimally coupled to gravity is governed by the action

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big] = S_{\rm EH} + S_\phi \tag{2.2}$$

The action is the sum of the gravitational Einstein-Hilbert action, $S_{\rm EH}$, and the action of a scalar field with canonical kinetic term, S_ϕ. The potential $V(\phi)$ describes the self-interactions of the scalar field. The energy-momentum tensor for the scalar field is given by

$$T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\big(\frac{1}{2}\partial^\sigma\partial_\sigma\phi + V(\phi)\big) \tag{2.3}$$

and the field equation of motion by

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) + V_{,\phi} = 0 \tag{2.4}$$

where $V_{,\mu} = \frac{dV}{d\phi}$. Assuming the FLRW metric for $g_{\mu\nu}$ and restricting to the case of a homogeneous field $\phi(t,\mathbf{x}) \equiv \phi(t)$, the scalar energy-momentum tensor takes the form of a perfect fluid, such that

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.5}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.6}$$

The resulting equation of state

$$\omega_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} \tag{2.7}$$

shows that a scalar field can lead to negative pressure ($\omega_\phi < 0$) and accelerated expansion ($\omega_\phi < -\frac{1}{3}$) if the potential energy V dominates over the kinetic energy $\frac{1}{2}\dot{\phi}^2$. The dynamics of the homogeneous scalar field and the FLRW geometry is determined by the equations of motion, which are found to be

$$H^2 = \frac{1}{3M_{Pl}^2}\Big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big), \qquad \dot{H} = -\frac{\dot{\phi}^2}{2M_{Pl}^2}, \qquad \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \tag{2.8}$$

where the first equation is the Friedman equation, the second equation is the continuity equation, and the third equations describes the evolution of the inflaton.

For large values of the potential, the field experiences significant Hubble friction for the term $H\dot{\phi}$.

2.2 Slow-roll inflation

We begin by deriving the acceleration equation for a universe dominated by a homogeneous scalar field in the slow-roll case as

$$\frac{\ddot{a}}{a} == \dot{H} + H^2 = -\frac{1}{6}(\rho_\phi + 3p_\phi) = H^2(1-\epsilon) \tag{2.9}$$

where

$$\epsilon = \frac{3}{2}(\omega_\phi + 1) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \tag{2.10}$$

Notice that in the de Sitter limit, $p_\phi \to -\rho_\phi$ corresponds to $\epsilon \to 0$. In this case, the potential energy dominates over the kinetic energy, $\dot{\phi}^2 \ll V(\phi)$.

The first slow-roll parameter ϵ may be related to the evolution of the Hubble parameter H such that

$$\epsilon = -\frac{\dot{H}}{H^2} = -\frac{\mathrm{d}\ln H}{\mathrm{d}N} \tag{2.11}$$

where $\mathrm{d}N = H\mathrm{d}t$. Accelerated expansion occurs if $\epsilon < 1$; as soon as $\epsilon < 1$ fails (i.e. when slow-roll conditions are violated), inflation ends. Accelerated expansion will only be sustained for a sufficiently long period of time if the second time derivative of ϕ is small enough $|\ddot{\phi}| \ll |3H\dot{\phi}|, |V_{,\phi}|$.

This requires smallness of a second slow-roll parameter η of the form

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon - \frac{1}{2\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{d}N} \tag{2.12}$$

where $|\eta| < 1$ ensures that the fractional charge of ϵ per e-fold is small.

Alternatively, note that to ensure that inflation lasts for at least 60 $e-$folds, the potential is required to be very flat, flat enough such that the slow-parameters $\epsilon, \eta, \ldots$ are much less than 1 most of the time. Such slow-roll parameters are defined as the conditions by which inflation must go through at least $\mathcal{O}(60)$ $e-$foldings, i.e. to achieve this much inflation, the Hubble parameter H cannot change within a Hubble parameter cannot change within a Hubble time H^{-1} such that

$$\epsilon = -\frac{\dot{H}}{H^2} \ll \mathcal{O}(1), \qquad \eta \equiv \frac{\dot{\epsilon}}{\epsilon H} \ll \mathcal{O}(1) \tag{2.13}$$

The slow-roll conditions, $|\epsilon|, |\eta| < 1$, may also be expressed as conditions on the shape of the inflationary potential

$$\epsilon_V(\phi) \equiv \frac{M_{\rm Pl}^2}{2}\Big(\frac{V_{,\phi}}{V}\Big)^2, \qquad\qquad \eta_V(\phi) \equiv M_{\rm Pl}^2 \frac{V_{,\phi\phi}}{V} \tag{2.14}$$

where in this case the Planck mass to make ϵ_V and η_V manifestly dimensionless. In what follows we will set $M_{\rm Pl} = 1$ again. In the slow-roll regime, $\epsilon_V, |\eta_V| \ll 1$, the background evolution is $H^2 \approx \frac{1}{3}V(\phi) \approx$ constant, $\dot{\phi} \approx -\frac{V_{,\phi}}{3H}$, and the space-time is approximately de Sitter $a(t) \sim e^{Ht}$.

The parameters ϵ_V and η_V are called the potential slow-roll parameters to distinguish them from the Hubble slow-roll parameters ϵ and η. In the slow-roll approximation the Hubble and potential slow-roll parameters are related as follows.

There in fact exists a full hierarchy of slow-roll parameters [8] which are built from V and its derivatives $V_{,\phi}, V_{,\phi\phi}, \ldots$, etc. For example, we can define a second-order slow-roll parameter

$$\xi = M_{Pl}^4\Big(\frac{V_{,\phi}V_{,\phi\phi\phi}}{V^2}\Big) \tag{2.15}$$

which is related to the third derivative of the potential. At first order, slow-roll parameters ϵ and η may be considered constant as the potential is very flat, i.e.
$\dot{\epsilon}, \dot{\eta} = \mathcal{O}(\epsilon^2, \eta^2)$

$$\epsilon \approx \epsilon_V, \qquad \eta \approx \eta_V - \epsilon_V \tag{2.16}$$

Inflation ends when the slow-roll conditions are violated

$$\epsilon(\phi_{\rm end}) \equiv 1, \qquad \epsilon_V(\phi_{\rm end}) \approx 1 \tag{2.17}$$

In order for inflation to last long enough to actually solve the flatness and horizon problems, a small smooth patch small than the Hubble radius must grow to encompass *at least* the entire observable Universe. This is defined in terms of e-foldings: the number of e-folds before inflation ends is

$$N(\phi) \equiv \ln \frac{a_{\text{end}}}{a} = \int_{t}^{t_{\text{end}}} H dt = \int_{\phi}^{\phi_{\text{end}}} \frac{H}{\dot{\phi}} \approx \int_{\phi_{\text{end}}}^{\phi} \frac{V}{V_{,\phi}} d\phi \tag{2.18}$$

The result of which may be written as

$$N(\phi) = \int_{\phi_{\text{end}}}^{\phi} \frac{d\phi}{\sqrt{23}} \approx \int_{\phi_{\text{end}}}^{\phi} \frac{d\phi}{\sqrt{2\epsilon_V}} \tag{2.19}$$

To solve the horizon and flatness problems requires that the total number of inflationary e-folds exceeds about 60 such that

$$N_{\text{total}} = \ln \frac{a_{\text{end}}}{a_{\text{start}}} \gtrsim 60 \tag{2.20}$$

The precise value depends on the energy scale of inflation and on the details of reheating after inflation. The fluctuations observed in the CMB are created during approximately $N_{CMB} \approx 40 - 60$ e-foldings before the end of inflation. The following integral constraint gives the corresponding field value ϕ_{CMB}

$$\int_{\phi_{\text{end}}}^{\phi_{\text{CMB}}} \frac{d\phi}{\sqrt{2\epsilon_V}} = N_{\text{CMB}} \approx 40 - 60 \tag{2.21}$$

2.3 Categories of Inflation

Unfortunately, physicists remain at a loss regarding fundamental questions about the inflationary paradigm: What is the inflaton? Why did the universe begin at such a high energy state? What is the shape of the inflationary potential? What new physics drives inflation?

All of these questions remain unanswered.

Inflation may basically be divided into two categories upon the type of potential $V(\phi)$, which is related to the distance which the inflaton field moves, i.e. $\Delta\phi \equiv \phi_{\text{CMB}} - \phi_{\text{end}}$, as measured in Planck units:

- *Small field Inflation*

  Small-field models [9-11] have the field moving over a sub-Planckian distance, i.e. $\Delta\phi < M_{\text{Pl}}$, which is observationally relevant as small-field models predict the amplitude of gravitational waves too small to be detected.

To offer a simple example of such a potential for small field inflation, observe the Higgs potential [12]

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu} \right)^2 \right]^2 \tag{2.22}$$

which relies upon spontaneous symmetry breaking such that the field rolls off an unstable equilibrium point toward a displaced vacuum to achieve small-field evolution.

- *Large-Field inflation*

  Here the inflaton field begins at large-field values and then evolves to a minimum at the origin $\phi = 0$. In this case the field evolution is super-Planckian, i.e. $\Delta\phi > M_{\rm pl}$, and therefore gravitational waves produced by inflation should be observable. Experiments in the near future are expected to have the capability to detect such a gravitational wave background (more on this later).

  A quintessential example for a large-field inflationary potential is given by "chaotic inflation" [13]

$$V(\phi) = \lambda_p \phi^p \tag{2.23}$$

  where the coupling constant λ_p is independent of slow-roll parameters, and must be set very small $\lambda_p \lll 1$ to achieve the correct density fluctuations observed. Furthermore, note for super-Planckian values $\Delta\phi \ggg M_{\rm Pl}$, slow-parameters remain very small.

2.4 Inflationary Scenarios

Now let's begin to consider to the myriad of inflationary models available, a result of theorists struggling to understand inflationary dynamics and the implications of such physics. Note that inflation is more of a framework, certainly not a unique theory, and therefore theorists can achieve such inflationary expansion in a multitude of ways, usually by writing down an action and then arguing that such a theory is theoretically and/or phenomenologically well-motivated. However, it would be counterproductive to rigorously detail the landscape of inflationary scenarios (we recommend [14] to interested readers). As an introduction to inflationary models beyond the simplest single-field slow-roll scenario, we present a few broad classifications of inflationary cases:

- *Non-canonical Kinetic Terms*

  In our simple slow-roll case, we used the canonical kinetic term such that

$$\mathcal{L}_\phi = X - V(\phi), \quad \text{where} \;\; X \equiv \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{2.24}$$

  Observe that we could easily extend this to include fields with non-canonical terms such that

$$\mathcal{L}_\phi = F(\phi, X) - V(\phi) \tag{2.25}$$

  where the function $F(\phi, X)$ denotes derivatives of the inflaton field. A motivation of such non-canonical terms relates to the potential $V(\phi)$: in our canonical slow-roll scenario, inflation occurs only if the potential is very flat; non-canonical terms can drive inflation even with a very steep potential.

- *Multifield Inflation*

  We may extend the number of inflationary fields, which affects the dynamics of inflation and the mechanism to produce fluctuations in countless ways. Indeed, a downside to such an approach is that it becomes cumbersome to extract predictions from many such models.

- *Gravity*

  The vanilla action in eq. (2.2) is minimally coupled to gravity, i.e. there exists no direct coupling between the metric and the inflaton field. Consequently, we could easily construct models with a non-minimal coupling between the graviton and inflaton. Furthermore, perhaps the Einstein-Hilbert term S_{EH} may be UV-modified via $f(R)$ theories (or affected by any other possible modified gravity scenario currently discussed).

2.5 Perturbations During Inflation

Before continuing to classify inflationary models, we will review the basic mechanism of quantum fluctuations from the inflaton field $\delta\phi(t, \mathbf{x})$.

(Readers are recommended to consult the background of CMB physics in Appendix A and cosmological perturbation theory basics in Appendix B before continuing with this section.)

Statistical Properties of Cosmological Fluctuations

Let's review the basic statistics of cosmological perturbations. Cosmologists normally characterize the properties of a perturbation field in terms of the power spectrum. As a general definition, we introduce a random field $\mathcal{R}(\mathbf{x})$. As we are working in flat space, we can Fourier transform $\mathcal{R}(\mathbf{x})$ such that

$$\mathcal{R}_{\mathbf{k}} = \int d^3x \mathcal{R}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{2.26}$$

and define the dimensionless power spectrum $P_{\mathcal{R}}$ to be

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_{\mathcal{R}}(k), \qquad \Delta^2_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) \tag{2.27}$$

where the brackets $\langle \ldots \rangle$ denote ensemble averaging of (in our case) the fluctuations. This measures the amplitude of the fluctuations at a given scale k_i. (Note that we have chosen the normalization of the dimensionless power spectrum $\Delta^2_{\mathcal{R}}(k)$ such that the variance of $\mathcal{R}$ is $\langle \mathcal{R}\mathcal{R} \rangle = \int_0^\infty \Delta^2_{\mathcal{R}}(k) d\ln k$. Moreover, throughout the rest of the paper $\mathcal{R}$ will denote the comoving curvature perturbation as detailed in Appendix A).

For the purposes of our paper, it is crucial to recognize that if the fluctuations are exactly Gaussian, then the primordial power spectrum $P_{\mathcal{R}}(k)$ contains all the information possible. Furthermore, observe that the power spectrum is a two-point correlation function. What are we actually calculating when we compute the power spectra of primordial fluctuations? To provide some motivation, consider QFT as used high-energy particle physics: here, two-point correlation functions of fields describe freely propagating particles in Minkowski spacetime. Particle colliders are built to probe the more interesting properties of these fields, i.e. to measure their higher order correlations. With the assumption that quantum fluctuations sourced cosmological perturbations during inflation, we interpret the power spectra of primordial fluctuations as describing freely propagating particles in the inflationary background. Higher-order correlations probe the details of inflationary dynamics (interactions, broken degeneracies, etc.). Therefore, measuring such higher-order correlations could allow us to differentiate between and falsify inflationary models! (We will more thoroughly discuss this in Chapter 3. See Figure 2.1 for a schematic explanation of the power spectra in relation to inflation.)

Allow us to continue detailing the basic statistics of fluctuations: we can also measure the scale-dependence (or slope) of the power spectrum by defining the scale spectral index (or tilt) $n_f(k)$ such that

$$n_f - 1 \equiv \frac{d \ln \Delta_s^2}{d \ln k} \tag{2.28}$$

where scale-invariance corresponds to the value $n_f = 1$. By defining the running of the spectral index with α_f

$$\alpha_f \equiv \frac{d n_s}{d \ln k} \tag{2.29}$$

we may approximate the power spectrum by a power law in the form

$$\Delta_f^2(k) = A_f(k_*) \Big(\frac{k}{k_*} \Big)^{n_f(k_*) - 1 + \frac{1}{2}\alpha_f(k_*) \ln(k/k_*)} \tag{2.30}$$

where k_* is an arbitrary reference scale.

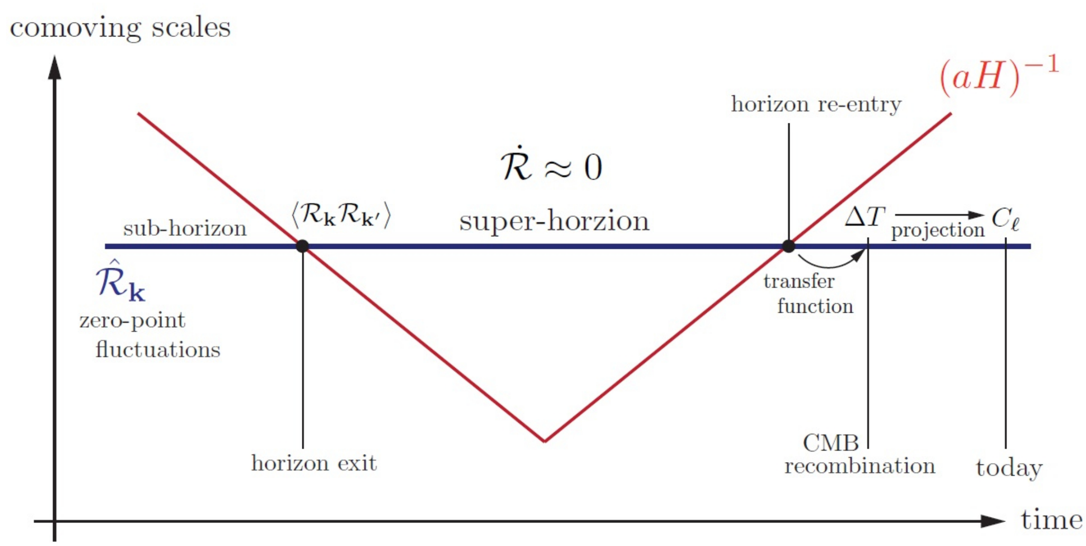

Figure 2.1: The above diagram of inflation depicts how inflation solves the horizon problem and elucidates how perturbations were created and evolved in the early universe: Scales of cosmological interest were far smaller than the Hubble radius at very early times pre-inflation and re-entered the Hubble radius at very late times post-inflation. Recall the comoving Hubble radius $\big(aH\big)^{-1}$ shrinks during inflationary expansion while comoving scales k^{-1} scales remain constant. Thus perturbations exit the horizon, are frozen until horizon re-entry (as causal physics do not effect perturbations on super-horizon scales, $\dot{\mathcal{R}} \approx 0$), and subsequently result in CMB anistropies and perturbations which form LSS. Thus, our goal is to match observed anistropies C_l to the predicted power spectrum $P_{\mathcal{R}}$ at horizon exit.

Scalar Perturbations

In the case of scalar (or density) perturbations in inflation, let's begin by choosing a gauge such that the energy density of the inflaton field is unperturbed, $\delta\rho_\phi = 0$. Using this, all scalar degrees of freedom are expressed by the metric perturbation $\zeta = (t, \mathbf{x})$

$$g_{ij} = a^2(t)[1 + 2\zeta]\delta_{ij} \tag{2.31}$$

where ζ is a measure of spatial curvature of constant-density hypersurfaces $\mathcal{R}^{(3)} = -4\nabla^2\Psi/a^2$ which remains constant outside the horizon (for adiabatic perturbations). By taking into account the transfer functions describing the sub-horizon evolution of fluctuations, the primordial value of ζ may be related to observations from the CMB and LSS. What follows is a power spectrum of ζ

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} P_s(k) \tag{2.32}$$

with similar definitions of the tilt, running, and approximate power spectrum as above.

We stress again: if ζ is Gaussian, then the power spectrum contains all statistical information, but it is possible that primordial non-Gaussianity is encoded in higher-order correlation functions of ζ, which may differentiate between inflationary models (i.e. between very small non-Gaussianity of single-field slow-roll, to significant non-Gaussianity of non-trivial single-field models, multifield models, or violation of slow-roll conditions).

Tensor Perturbations

In the case of tensor perturbations corresponding to gravitational wave fluctuations, we use a gauge-invariant metric perturbation h_{ij}

$$g_{ij} = a^2(t)[\delta_{ij} + h_{ij}], \qquad \partial_j h_{ij} = h^i_i = 0 \tag{2.33}$$

We define the power spectrum for the two polarization modes of h_{ij} as

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} P_t(k) \tag{2.34}$$

where $h_{ij} = h^+ e^+_{ij} + h^\times e^\times_{ij}, \quad h \equiv h^+,\ h^\times$.

Defining the power spectrum of tensor perturbations as the sum of the power spectra for the two polarizations

$$\Delta_t^2 \equiv 2\Delta_h^2 \tag{2.35}$$

Analogously, we define the scale-dependence as

$$n_t \equiv \frac{d \ln \Delta_t^2}{d \ln k} \tag{2.36}$$

(where for historical reasons it is written without -1). This gives us an approximate $\Delta_t^2(k)$ given as

$$\Delta_t^2(k) = A_t(k_*)\Big(\frac{k}{k_*}\Big)^{n_t(k_*)} \tag{2.37}$$

As detailed in Appendix C, the parameter r is given as

$$r \equiv \frac{\Delta_t^2}{\Delta_h^2} \tag{2.38}$$

This ratio of tensor power to scalar power is sensitive to CMB polarization measurements.

2.6 Primordial Power Spectra

Now we can summarize the results of the primordial power spectra for scalar and tensor fluctuations, cosmological observables which allows us to constrain (and falsify) various inflationary models. Although the previous section provided a modest sketch of these results in terms of statistical properties, we fully derive these results in Appendix C.

The power spectrum of ζ and the power spectrum of inflation fluctuations $\delta\phi$ are related as follows

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = \left(\frac{H}{\dot{\phi}}\right)^2 \langle \delta\phi_{\mathbf{k}} \delta\phi_{\mathbf{k}'} \rangle \tag{2.39}$$

For slow-roll inflation, quantum fluctuations of a light scalar field (i.e. $m_\phi \ll H$) in quasi-de Sitter space (i.e. $H \approx$ const.) scale with the Hubble parameter H such that

$$\langle \delta\phi_{\mathbf{k}} \delta\phi_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} \left(\frac{H}{2\pi}\right)^2 \tag{2.40}$$

where the right hand side of eq. (2.40) is to be evaluated at $k = aH$, the horizon exit of a given perturbation.

Inflationary quantum fluctuations produce following power spectrum for ζ

$$P_s(k) = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=aH} \tag{2.41}$$

Quantum fluctuations during inflation excite tensor metric perturbations h_{ij}. Their power spectrum is simply that of a massless field in de Sitter space

$$P_t(k) = \frac{8}{M_{\rm Pl}^2} \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=aH} \tag{2.42}$$

Summary of Slow-roll Predictions

(We again advise readers to consult Appendix C for details.)

Models of single-field slow-roll inflation makes definite predictions for the primordial scalar and tensor fluctuation spectra. Under the slow-roll approximation one may related the predictions for $P_s(k)$ and $P_t(k)$ to the shape of the inflaton potential $V(\phi)$. To compute the spectral indices one uses $d\ln k \approx d\ln a$ where ($H \approx$ constant). To first order in the slow-roll parameters ϵ and η one finds

$$P_s(k) = \frac{1}{24\pi^2 M_{\rm Pl}^4} \frac{V}{\epsilon}\Bigg|_{k=aH}, \quad \text{where} \qquad n_s - 1 = 2\eta - 6\epsilon \tag{2.43}$$

$$P_t(k) = \frac{2}{3\pi^2} \frac{V}{M_{\rm Pl}^4}\Bigg|_{k=aH}, \quad \text{where} \qquad n_t = -2\epsilon, \quad r = 16\epsilon \tag{2.44}$$

We note that the value of the tensor-to-scalar ratio depends on the time-evolution of the inflation field, such that

$$r = 16\epsilon = \frac{8}{M_{\rm Pl}^2}\left(\frac{\dot{\phi}}{H}\right)^2 \tag{2.45}$$

We also point out the existence of a slow-roll consistency relation between the tensor-to-scalar ratio and the tensor tilt which, at lowest order, has the form

$$r = -8n_t \tag{2.46}$$

2.7 Current Observational Constraints from WMAP

Using WMAP 5-year temperature and polarization data [15], we present current observational constrains on the primordial spectra $P_s(k)$ and $P_t(k)$. Furthermore, we can refine these constraints with data of the angular diameter distance of Baryon Acoustic Oscillations (BAO) [16] at $z = 02$ and 0.35, and data of the luminosity distance of Type Iα Supernovae (SN) [17] at $z \leq 17$.

Parameter	**5-yr WMAP**	**WMAP+BAO+SN**
n_s	$0.963^{+0.014}_{-0.015}$	$0.960^{+0.013}_{-0.013}$
n_s	0.986 ± 0.022	0.970 ± 0.015
r	< 0.43	< 0.22
n_s	$1.031^{+0.054}_{-0.055}$	$1.017^{+0.042}_{-0.043}$
α_s	-0.037 ± 0.028	$-0.028^{+0.020}_{-0.020}$
n_s	$1.087^{+0.072}_{-0.073}$	$1.089^{+0.070}_{-0.068}$
r	< 0.58	< 0.55
α_s	-0.050 ± 0.034	-0.058 ± 0.028

Table 2.1: Results of 5-year WMAP constraints for $(n_s), (n_s, r), (n_s, \alpha_s)$, and (n_s, r, α) as used in the power-law parametrization of the power spectrum using both WMAP data and WMAP data combined with SN and BAO data.[15,207]

Using the power-law parametrization of the scalar power spectrum at horizon crossing $k_* = a(t_*)H(t_*)$ of the form

$$P_s(k) = A_s(k_*)\Big(\frac{k}{k_*}\Big)^{n_s(k_*)-1+\frac{1}{2}\alpha_s(k_*)\ln(k/k_*)} \tag{2.47}$$

we find the amplitude of scalar fluctuations at $k_* = 0.002\ \mathrm{Mpc}^{-1}$

$$A_s = (2.445 \pm 0.096) \times 10^{-9} \tag{2.48}$$

and the scale-dependence of the power spectrum

$$n_s = 0.960 \pm 0.013 \tag{2.49}$$

where we've assumed no tensors, i.e. $(r \equiv 0)$. Note that the Harrison-Zel'dovich-Peebles spectrum (i.e. scale-invariant spectrum) $n_s = 1$ is 3.1 standard deviations away from the mean of the likelihood.

If we consider the possibility of $r \neq 0$, we calculate the upper bound to be

$$r < 0.22 \quad (95\% \text{ CL}) \tag{2.50}$$

using data from temperature $\langle TT \rangle$ and temperature-polarization $\langle TE \rangle$ cross-correlation measurements. With a non-zero r, $r \neq 0$, our constraint on n_s becomes

$$n_s = 0.970 \pm 0.015 \tag{2.51}$$

Furthermore, WMAP has detected no evidence for curvature ($-0.0179 < \Omega_k < 0.0081$), running ($-0.068 < \alpha_s < 0.012$), or isocurvature.

2.8 Inflationary Model Building

Now we are prepared to present an overview of the varieties of inflationary models and their associated phenomenology. What's the standard approach to all this? Well, theorists usually write down some Lagrangian (normally with one or many scalar fields) and make arguments about how natural or well-motivated or consistent this theory is. Then, we try to extract predictions from these theories and test them (e.g. via a detection of a gravitational wave background).

Another approach is to take observations, and present a model based on those considerations. We'll discuss one such model in section 2.83.

2.8.1 Single-Field Slow-Roll

We have introduced slow-roll single-field previously as a canonical scalar field ϕ minimally coupled to gravity

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \big[R - (\nabla\phi)^2 - 2V(\phi) \big] \tag{2.52}$$

A measurement of the amplitude and the scale-dependence of the scale and tensor spectra directly constrains the shape of the inflaton potential $V(\phi)$. That is, by normalizing the potential on CMB scale such that $v(\phi) \equiv V(\phi)/V(\phi_{\rm CMB})$, the parameters r and n_s become

$$r = 8(v')^2\big|_{\phi=\phi_{\rm CMB}}, \quad n_s - 1 = \big[2v'' - 3(v')^2\big]\big|_{\phi=\phi_{\rm CMB}} \tag{2.53}$$

and thus determine the shape of the inflaton potential (v', v'') at $\phi_{\rm CMB}$.

2.8.2 Beyond slow-roll single field

We have also discussed previously several ways to construct inflationary theories beyond simple single-field slow roll. We will categorize these models as follows:

- *Non-trivial kinetic terms*

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\big[R + 2P(X, \phi, \partial\phi, \partial^2\phi, \ldots)\big] \tag{2.54}$$

  such that $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and slow-roll is given by the special case $P(X, \phi) = X - V(\phi)$.

  Examples of such models with non-canonical terms include K-inflation [18], DBI inflation [19], and Ghost inflation [20, 21].

  Writing $P(X, \phi)$ as the pressure of the scalar fluid, with energy density $\rho = 2XP_{,X} - P$, these models may characterized by the sound speed given by

$$c_s^2 \equiv \frac{P_{,X}}{\rho_{,X}} = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}} \tag{2.55}$$

- *Multifield Inflation*

  Additionally we may employ two or more scalar fields during inflation [22-25]. As we've previously discussed, the major drawback of multifield models is that their predictive power is often fuzzy. However, the "smoking gun" for more than one scalar field at during inflation would be the detection of non-adiabatic perturbations.

  Some multifield models produce features in the spectrum of adiabatic perturbations [26-33] , seeding isocurvature perturbations which could leave imprints on the CMB [34-39] (see more information in Appendix A).

  To provide some intuition into the array of predictions for inflationary models with more than one scalar field, we will detail several classes of inflationary models which decouple the creation of density perturbations from the inflaton dynamics during inflation:

  For "inhomogeneous heating" models [40-41], when the decay of the vacuum energy at the end of inflation is sensitive to the local values of fields other than the inflation, primordial perturbations may be generated from such so-called "inhomogeneous heating" or via "modulated hybrid inflation" [42].

In "curvaton" models, density perturbations are created via the inhomogeneous distribution of a weakly coupled field (i.e. the "curvaton") when the field decays into radiation at some time after inflation. In fact, we mention such a model as it predicts the existence of isocurvature density perturbations, which may be produced in some particle species (e.g. baryons) whose abundance differs from the thermal equilibrium abundance when the curvaton decays.

- *Models of Non-Standard Vacuum and Time-Varying Sound Speeds*

  In the next chapter, we will detail inflationary models which begin in a non-standard vacuum, such that inflation may have begun in an excited state rather than the standard Bunch-Davis vacuum [47]. Furthermore, notice we may be able to write consistent inflationary models with the sound speed of the fluctuations c_s^2 in eq. (2.55) in the superluminal limit [48-51].

2.8.3 Observationally Inspired Inflationary Models

As opposed to the previous procedure of connecting theoretical models to observations, let's take the opposite approach and discuss one such inflationary model which is primarily justified from phenomenological considerations [52-53]:

It has been consistently found that the WMAP temperature $\langle TT \rangle$ power spectrum contains an anomalous dip at $l \sim 20$ and a bump at $l \sim 40$ [54-59]. One explanation for this structure is the presence of phenomenological feature in the primordial curvature power spectrum, which may be a relic from inflation. (Otherwise, it may just be a statistical anomaly.) Such power spectrum features could arise when slow-roll conditions are momentarily violated. We may model such an effect by considering it caused through a "step-like" feature in the inflationary potential [60].

In order to test whether the $l \sim 20 - 40$ anomaly is indeed a result of inflation, we adopt a phenomenological inflationary potential of the form $V(\phi) = m^2_{\rm eff}(\phi)\phi^2/2$ where the effective mass of the inflaton ϕ has a step at $\phi = b$ corresponding to the sudden change in mass during a phase transition, such that

$$m^2_{\rm eff}(\phi) = m^2 \left[1 + c \tanh\left(\frac{\phi - b}{d} \right) \right] \tag{2.56}$$

Note that we began by considering the simplest inflaton potential $V(\phi) = \frac{1}{2}m^2\phi^2$ and then imposed a tanh fluctuation to give a step potential. Observe that the amplitude and width of the step are determined by c and d respectively, where b and m are potential parameters in units of Planck mass.

We may be able to test the $l \sim 20-40$ hypothesis via large-scale polarization of the CMB, as such a feature in the inflationary potential results in an $E-$mode polarization spectrum similar to that in the temperature power spectrum. By deriving the slow-roll parameters in this model, we may use current and future CMB surveys to constrain the parameters of the inflationary potential.

A potential downside to this approach is that due to the parameters of the inflationary potential, each of the possible models must generally be computed on a case-by-case basis. However, the Planck satellite is projected to be of the statistical sensitivity to constrain such models at a 3σ significance level, thus confirming or falsifying this effect. Moreover, it is possible that such an inflationary potential would in fact have a detectable size of non-Gaussianity [61]. For more information, see [62].

2.9 Future Tests of Inflation

In some sense, although inflation has past generic tests (i.e. flatness of the universe and near scale-invariance, Gaussianity, and adiabaticity of the density fluctuations) as observations have become more and more precise, we've only just begun to test and refine the inflationary paradigm itself. As our experiments are becoming sensitive enough to distinguish between inflationary models and alternatives, we may soon learn whether these generic predictions remain valid, or have just provided us with a false-sense of confidence for all these years.

- **Primordial Tensor Modes**

  As detailed in Appendix A, detecting primordial $B-$modes would considered a clear signature for inflationary gravitational waves, i.e. detecting C^{BB} with gives us $P_h(k)$ with a tensor amplitude A_t

$$\Delta_t^2(k) \equiv \frac{k^3}{2\pi^2} P_h(k) = A_t \left(\frac{k}{k_*}\right)^{n_t}, \quad n_t = -2\epsilon \approx 0 \tag{2.57}$$

  Under normal assumptions (see recent paper [63] as detailed in Appendix A), measuring A_t provides direct access to the energy scale of inflation. Indeed, a robust detection of a gravitational wave signal would point towards inflationary models with a large vacuum energy, i.e. $V^{1/4} \gtrsim 10^{16}$ GeV. Such a detection would in fact support the simplest slow-roll scenario, as a significant gravitational wave background is usually negligible for inflationary models where light fields lay density perturbations, and alternatives such as the ekpyrotic scenario [64] and VSL theories [48].

- **Scale Dependence of Modes**

  In the case of scalar modes, we concentrate on the scale dependence of scalar modes via variation the spectral index n_s (or the running), denoted by α_s such that

$$\alpha_s \equiv \frac{dn_s}{d\ln k} \tag{2.58}$$

  This could either confirm (i.e. with a small running $\alpha_s \sim \mathcal{O}(\epsilon^2)$ is expected in slow-roll), or question to our conceptions of inflation and the generation of perturbations (i.e. with a large positive $|\alpha_s| > 0.001$ or even negative running).

  Furthermore for tensor modes, a single-field slow-roll consistency relation between tensor-to-scalar ratio r and the tensor spectral index n_t is of the form

$$r = -8n_t \tag{2.59}$$

  The possibility of $r \neq -8n_t$ provides another test for violations of slow-roll conditions.

- **Isocurvature Fluctuations**

  In general, there can be relative perturbations modes between different components (such as between radiation and matter)

$$S_m \equiv 3H\Big(\frac{\delta\rho_\gamma}{\dot{\rho}_\gamma} - \frac{\delta\rho_m}{\dot{\rho}_m}\Big) = \frac{\delta\rho_m}{\rho_m} - \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} \tag{2.60}$$

  Isocurvature density perturbations are scalar modes and cannot produce $B-$mode polarization. However, $E-$ mode polarization and the cross-correlation between temperature anisotropies and $E-$mode polarization can discriminate between isocurvature modes and purely adiabatic spectra with similar temperature power spectrum. Additional light scalar fields during inflation lead to additional non-adiabatic perturbations being frozen-in on large scales during inflation.

  The detection of isocurvature fluctuations is seen as a signature of multifield inflation, though exact theoretical predictions for the amplitude of such isocurvature perturbations are complicated due to its dependence on multifield inflationary dynamics *and* post-inflationary evolution.

Current constraints are given via a correlation parameter β

$$\beta \equiv \frac{P_{SR}}{\sqrt{P_S P_R}} \tag{2.61}$$

such that P_S, P_R are the power spectra of isocurvature and adiabatic fluctuations respectively with cross-correlations P_{SR}. We can therefore parametrize the relative amplitude between the two types of perturbations with coefficient α such that

$$\frac{P_S}{P_R} \equiv \frac{\alpha}{1-\alpha} \tag{2.62}$$

Presently, constraints on a possible isocurvature contribution are:

$$\alpha < 0.067 \quad \text{at } 95\% \text{ CL} \quad \text{for } \beta = 0 \tag{2.63}$$

$$\alpha < 0.0037 \quad \text{at } 95\% \text{ CL} \quad \text{for } \beta = -1 \tag{2.64}$$

where the uncorrelated case is denoted by $\beta = 0$ and the anti-correlated case denoted by $\beta = -1$.

Obviously the above diagnostics will be crucial for confirming simple single-field slow-roll inflation, or disfavoring the scenario in favor of the multifield case or something else altogether. In the next chapter, we detail another powerful probe for understanding the inflationary: deviations of Gaussian primordial fluctuations. We will detail how such a measurement could differentiate and even falsify inflationary models.

Chapter 3

Non-Gaussianity

3.1 Primordial Non-Gaussianity

We previously mentioned that non-Gaussian contributions to the correlations of cosmological fluctuations directly measure inflationary dynamics (e.g. the inflaton interactions). This allows us to put constraints on inflationary models and their alternatives, permitting us to differentiate between and constrain the litany of models available. Similarly, non-Gaussianity may also be used to constrain and falsify/confirm alternatives to the inflationary paradigm. Therefore, detecting non-Gaussianity of primordial fluctuations may be one of the most powerful probes we have of the early universe.

In Appendix C, we expanded the inflationary action to second order in $\mathcal{R}$, the comoving curvature perturbation. This allowed us to compute the power spectrum $P_{\mathcal{R}}(k)$. If the fluctuations $\mathcal{R}$ are drawn from a Gaussian distribution, then the power spectrum (or the two-point correlation function) contains all the information about the physics of inflation. Non-Gaussianity (expanding the action into the third order for leading non-trivial interaction terms) directly probes inflationary interactions, which will reveal much of inflationary dynamics.

We note that in the following section we shall compute three-point (and higher order) cosmological correlations; this is non-trivial due to the time-evolution of the vacuum in the presence of interactions. Refer to Appendix D where the "In-In" Formalism is reviewed. Moreover, because of the complexity of these computations, many of the following results will not be thoroughly derived. We direct interested readers to [67] for said derivations and more details.

3.1.1 The Bispectrum

The Fourier transform of the two-point function is given by the power spectrum

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 P_{\mathcal{R}}(k) \delta(\mathbf{k} + \mathbf{k}') \tag{3.1}$$

Then, using results via the "In-In" Formalism, we may analogously write the Fourier equivalent of the three-point function such that

$$\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \tag{3.2}$$

where the bispectrum $B_{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is subject only to momentum conversation via the delta-function. (Note that this is a result of translation invariance of the background, meaning the three vectors $\mathbf{k}_i$ form a closed triangle.)

Here the delta function (maintaining momentum conservation) is a consequence of translation invariance of the background. $B_{\mathcal{R}}$ is hence symmetric in its arguments and is a homogeneous function of degree -6 for scale-invariant fluctuations, i.e.

$$B_{\mathcal{R}}(\lambda \mathbf{k}_1, \lambda \mathbf{k}_2, \lambda \mathbf{k}_3) = \lambda^{-6} B_{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \tag{3.3}$$

Observe the number of independent variables is reduced to simply two via rotational invariance, e.g. the two ratios k_s/k_1 and k_3/k_1.

3.1.2 Local Non-Gaussianity

Cosmologists first parametrized non-Gaussianity phenomenologically with a non-linear correction to a Gaussian perturbation $\mathcal{R}_g$ known as "local non-Gaussianity" as it is local in real space [68]. Experimental constraints are normally set on the parameter $f_{\mathrm{NL}}^{\mathrm{local}}$, which is defined via

$$\mathcal{R}(\mathbf{x}) = \mathcal{R}_g(\mathbf{x}) + \frac{3}{5} f_{\mathrm{NL}}^{\mathrm{local}} \left[\mathcal{R}_g(\mathbf{x})^2 - \langle \mathcal{R}_g(\mathbf{x})^2 \rangle \right] \tag{3.4}$$

This relation was first derived in terms of the Newtonian potential $\Phi(\mathbf{x})$, such that $\Phi(\mathbf{x}) = \Phi_g(\mathbf{x}) = f_{\mathrm{NL}}^{\mathrm{local}} \left[\Phi_g(\mathbf{x})^2 - \langle \Phi_g(\mathbf{x})^2 \rangle \right]$. In cosmology, this is related to $\mathcal{R}$ in the matter era via a factor 3/5. Thus, note that this factor 3/5 is purely conventional.

Deriving the bispectrum of local non-Gaussianity from eq. (3.4), we find

$$B_{\mathcal{R}}(k_1, k_2, k_3) = \frac{6}{5} f_{\mathrm{NL}}^{\mathrm{local}} \times \left[P_{\mathcal{R}}(k_1) P_{\mathcal{R}}(k_2) + P_{\mathcal{R}}(k_2) P_{\mathcal{R}}(k_3) + P_{\mathcal{R}}(k_3) P_{\mathcal{R}}(k_1) \right] \tag{3.5}$$

For a scale-invariant spectrum $P_{\mathcal{R}}(k) = ak^{-3}$ and (without loss of generality) an ordering of the momentum such that $k_3 \leq k_2 \leq k_1$, this is given by the form

$$B_{\mathcal{R}}(k_1, k_2, k_3) = \frac{6}{5} f_{\mathrm{NL}}^{\mathrm{local}} \times A^2 \left[\frac{1}{(k_1 k_2)^3} + \frac{1}{(k_2 k_3)^3} + \frac{1}{(k_3 k_1)^3} \right] \tag{3.6}$$

It should be observed that the bispectrum for local non-Gaussianity is largest in the so-called "squeezed limit", i.e. when the smallest k (here denoted k_3) is very small, $k_3 \ll k_1 \sim k_2$. The other two momenta are then nearly equal. In the case of the squeezed limit, the bispectrum for local non-Gaussianity becomes

$$\lim_{k_3 \ll k_1 \sim k_2} B_{\mathcal{R}}(k_1, k_2, k_3) = \frac{12}{5} f_{\mathrm{NL}}^{\mathrm{local}} \times P_{\mathcal{R}}(k_1) P_{\mathcal{R}}(k_3) \tag{3.7}$$

Before continuing, let's pause to consider how we would relate the primordial bispectrum above to the CMB bispectrum. Recall we can related CMB temperature anisotropies a_{lm} to perturbations $\mathcal{R}$ via a transfer function $\Delta_l(k)$ through the integral

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \Delta_l(k) \mathcal{R}_{\mathbf{k}} Y_{lm}(\hat{\mathbf{k}}) \tag{3.8}$$

where Y_{lm} are eigenfunctions of the spherical harmonics.

The CMB bispectrum may be written as a three-point correlation of a_{lm} such that

$$B^{l_1 l_2 l_3}_{m_1 m_2 m_3} = \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle \tag{3.9}$$

and via substitution, we find the relation between primordial and CMB bispectra to be

$$\begin{aligned} B^{l_1 l_2 l_3}_{m_1 m_2 m_3} = (4\pi)^3 (-i)^{l_1+l_2+l_3} \int & \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} \Delta_{l_1}(k_1)) \Delta_{l_2}(k_2) \Delta_{l_3}(k_3) \\ & \langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \rangle Y_{l_1 m_1}(\hat{\mathbf{k}}_1) Y_{l_2 m_2}(\hat{\mathbf{k}}_2) Y_{l_3 m_3}(\hat{\mathbf{k}}_3) \end{aligned} \tag{3.10}$$

Although this technically is the correct form, there are many more computational methods required to actually analyze the CMB bispectrum and calculate possible non-Gaussianities (e.g. statistical estimators, Fisher matrices, etc.). See [69-70] for further details and background.

3.2 The Shape of Non-Gaussianities

Non-Gaussian bispectra contain a wealth of information about the physics of inflation. Recall that the Fourier modes $\mathbf{k}_i$ form a complete closed triangle (due to momentum conservation). As discussed in [71], this shape of non-Gaussianity relates to the different triangular configurations predicted via individual inflationary models (i.e. the mechanism which laid down primordial perturbations). We may now recognize the power of this signal: by measuring the shape of non-Gaussianity, we can therefore exclude inflationary models which predict unique shapes!

Let's begin by defining the shape function

$$S(k_1, k_2, k_3) \equiv N(k_1 k_2 k_3)^2 B_{\mathcal{R}}(k_1, k_2, k_3) \tag{3.11}$$

where $B_{\mathcal{R}}$ is the bispectrum and N is some normalization factor. The two most commonly cited shapes of non-Gaussianity are the "local" model, given by

$$S^{\text{local}}(k_1, k_2, k_3) \propto \frac{K_3}{K_{111}} \tag{3.12}$$

and the "equilateral" model, given by

$$S^{\text{equil}}(k_1, k_2, k_3) \propto \frac{\tilde{k}_1 \tilde{k}_2 \tilde{k}_3}{K_{111}} \tag{3.13}$$

where we've defined (following the notation by [72] to shorten exceptionally complex expressions):

$$\begin{aligned}
K_p =& \sum_i (k_i)^p \quad \text{with} \quad K = K_1 && (3.14)\\
K_{pq} =& \frac{1}{\Delta_{pq}} \sum_{i \neq j} (k_i)^p (k_i)^q && (3.15)\\
K_{pqr} =& \frac{1}{\Delta_{pqr}} \sum_{i \neq j \neq l} (k_i)^p (k_i)^q (k_l)^r && (3.16)\\
\tilde{k}_{ip} =& K_p - 2(l_i)^p \quad \text{with} \quad \tilde{k}_i = \tilde{k}_{i1} && (3.17)
\end{aligned}$$

such that $\Delta_{pq} = 1 + \delta_{pq}$ and $\Delta_{pqr} = \Delta_{pq}(\Delta_{qr} + \delta_{pr})$.

We previously noted that a scale-invariant fluctuations the bispectrum is only a function of the two ratios (e.g. k_2/k_1 and k_3/k_1), which prompts us to define the rescaled momenta $x_i = k_i/k_1$.

We may now categorize the various types of non-Gaussian triangles possible and relate them to inflationary models (and their alternatives). It has been found that often these models produce non-Gaussian perturbations with signals peaked in special triangle configurations. We categorize the major cases (note that we will discuss the details below, see Figure 3.1):

- *squeezed triangle* ($k_1 \approx k_2 >> k_3$)

  This is the dominant mode of models with multiple light fields during inflation. These include multifield inflation [73-85], the curvaton scenario [43,86], inhomogeneous heating models [40,41], and New Ekpyrotic models [87-93].

- *equilateral triangle* ($k_1 = k_2 = k_3$)

  Models with higher-derivative interactions result in signals which peak at equilateral triangles. These include models such as DBI inflation [19] and Ghost inflation [20,21], and more general models of non-trivial speeds of sound [94-97].

- *folded triangle* ($k_1 = 2k_2 = 2k_3$)

  Models with non-standard initial states result in folded triangle signals [47,94].

- For consistency, we also mention *elongated triangle* ($k_1 = k_2 + k_3$) and *isosceles triangles* ($d_1 > k_2 = k_3$), which are intermediate cases of the previous signals.

We should mention that we measure the magnitude of non-Gaussianity for arbitrarily shaped functions by defining the generalized $f_{\rm NL}$ parameter where the amplitude of non-Gaussianity is normalized in the equilateral configuration, given by the form

$$f_{\rm NL} \equiv \frac{5}{18} \frac{B_{\mathcal{R}}(k,k,k)}{P_{\mathcal{R}(k)^2}} \tag{3.18}$$

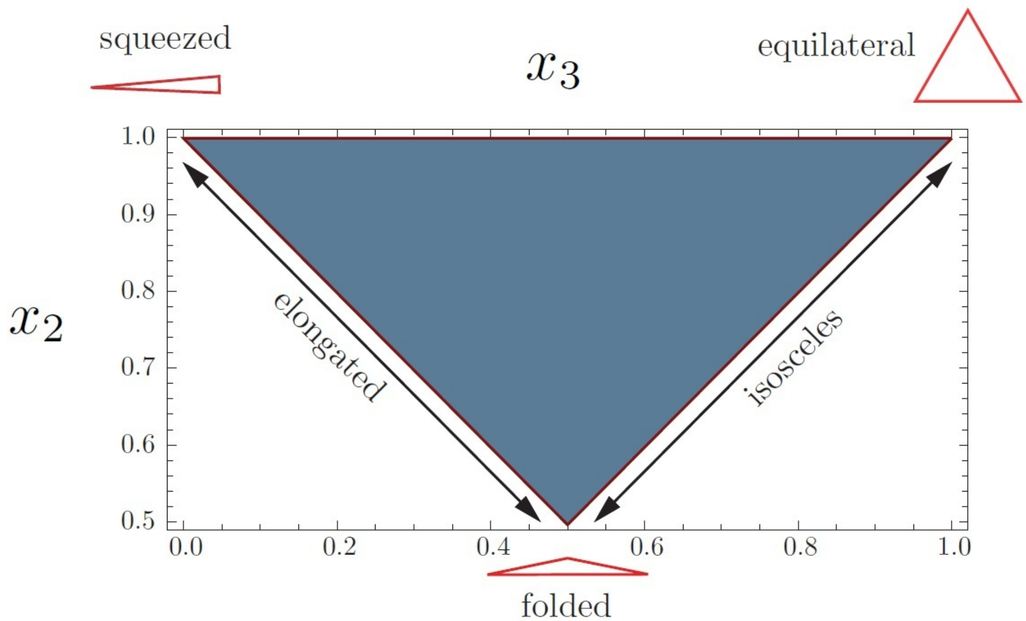

Figure 3.1: The various shapes of non-Gaussianity (squeezed, equilateral, folded, elongated, and isosceles) are depicted such that the triangles are parametrized by the rescaled momenta, $x_2 = k_2/k_1$ and $x_3 = k_3/k_1$. Note that the triangle inequality $x_2 + x_3 > 1$ is satisifed and the momenta are ordered $x_3 < x_2 < 1$. [12]

We summarize this section by writing the two non-Gaussian signals cosmologists usually concentrate on, $f_{\rm NL}^{\rm local}$ and $f_{\rm NL}^{\rm equil.}$, in Fourier space [69]:

- *Local Non-Gaussianity*, which is peaked on squeezed-triangles in Fourier space, where

$$F(k_1, k_2, k_3) = f_{\rm NL}^{\rm local} F_{\rm local}(k_1, k_2, k_3) \tag{3.19}$$

such that

$$F_{\rm local}(k_1, k_2, k_3) \sim \delta^{(3)}\Big(\sum_i k_i\Big)\left(\frac{1}{k_1^3 k_2^3} + \frac{1}{k_2^3 k_2^3} + \frac{1}{k_1^3 k_3^3}\right) \tag{3.20}$$

- *Equilateral Non-Gaussianity*, which is peaked on equilateral-triangles in Fourier space, where

$$F(k_1, k_2, k_3) = f_{\rm NL}^{\rm equil.} F_{\rm equil.}(k_1, k_2, k_3) \tag{3.21}$$

such that

$$F_{\text{equil.1}}(k_1, k_2, k_3) \sim \delta^{(3)}\Big(\sum_i k_i\Big)\left(\prod_{i=1} \frac{k_1 + k_2 + k_3 - 2k_i}{k_i^3}\right) \tag{3.22}$$

3.3 Theoretical Predictions

3.3.1 Single-Field Slow-Roll Bispectrum

It was found by Maldacena [99] that fluctuations due to single-field slow-roll inflation are expected to be extremely Gaussian, due to the weak interactions of the inflaton field. Normalizing S^{local} and S^{equil} such that $S^{\text{local}}(k,k,k) = S^{\text{equil}}(k,k,k)$, we may derive the bispectrum for slow-roll inflation to be

$$S^{\text{SR}}(k_1, k_2, k_3) \propto (\epsilon - 2\eta)\frac{K_3}{K_{111}} + \epsilon\Big(K_{12} + 8\frac{K_{22}}{K}\Big) \tag{3.23}$$

$$\approx (4\epsilon - 2\eta) S^{\text{local}}(k_1, k_2, k_3) + \frac{5}{3}\epsilon\ S^{\text{equil}}(k_1, k_2, k_3) \tag{3.24}$$

Here the bispectrum peaks at squeezed triangles and has an amplitude that is suppressed by slow-roll parameters

$$f_{\text{NL}}^{\text{SR}} = \mathcal{O}(\epsilon, \eta) \tag{3.25}$$

(Intuitively, we would expect such a result, as slow-roll parameters characterize deviations of the inflaton from a free field). As slow-roll parameters are of the order $\mathcal{O}(10^{-2})$, $f_{\text{NL}}^{\text{SR}}$ is expected to be $f_{\text{NL}}^{\text{SR}} \sim \mathcal{O}(10^{-2})$ for these models. This signal is undetectable for all current and foreseen experiments.

Moreover, we should note it has been calculated that even if inflation begins with such extremely Gaussian primordial perturbations, non-linear effects relating to CMB evolution will only generate a non-Gaussianity of order at most $f_{\text{NL}}^{\text{SR}} \sim \mathcal{O}(1)$ [98]. It seems unlikely that we will be able to detect such a signal in the near future. However, these effects must be more systemically characterized, as well will discuss at the end of this section.

3.3.2 Single-Field Inflation and the Maldacena Theorem

What about more general single-field inflationary models? For all single-field models (i.e. the only assumption being that the single inflaton is the only dynamical field during inflation), Creminelli and Zaldarriaga proved a consistency relation [99,100] stating that the three-point function in the squeezed limit is suppressed by $(1 - n_s)$, where $n_s - 1$ is the usual tilt of the scalar spectrum, thereby vanishing completely for scale-invariant perturbations. Therefore, detecting non-Gaussianity in the squeezed limit rules out vanilla single-field inflation! This result is valid irrespectively of any other assumptions about single-field inflation (e.g. the form of the potential, the type of kinetic term or sound speed, or the initial vacuum state).

Without loss of generality, let's consider a squeezed triangle signal, which correlates two short-wavelength modes, say $k_S = k_1 \approx k_2$, to one long-wavelength mode on long-wavelength mode, $k_L = k_3$. In the squeeze limit $k_1 \approx k_2 \ggg k_3$, we may write the theorem as

$$\lim_{k_3 \to 0} \langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \rangle = -(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)(n_s - 1) P_{\mathcal{R}}(k_1) P_{\mathcal{R}}(k_3) \tag{3.26}$$

where

$$\langle \mathcal{R}_{\mathbf{k}_i} \mathcal{R}_{\mathbf{k}_j} \rangle = (2\pi)^3 \delta(\mathbf{k}_i + \mathbf{k}_j) P_{\mathcal{R}}(k_i) \tag{3.27}$$

This consistency relation could possibly falsify simple single-field inflation.

Let's provide some motivation for the theorem:

Consider the squeezed triangle signal, which correlates two short-wavelength modes, $k_S = k_1 \approx k_2$, to one long-wavelength mode on long-wavelength mode, $k_L = k_3$. This results in a bispectrum

$$\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \rangle \approx \langle (\mathcal{R}_{\mathbf{k}_S})^2 \mathcal{R}_{\mathbf{k}_L} \rangle \tag{3.28}$$

Recall that during inflation, modes with longer wavelengths freeze earlier than modes with shorter wavelengths. Hence by definition, k_L will already be frozen outside the horizon when the two smaller modes freeze. Furthermore, it then acts a background field for the two short-wavelength modes.

Now, the theorem states that $(\mathcal{R}_{\mathbf{k}_S})^2$ is not correlated with $\mathcal{R}_{\mathbf{k}_L}$ if $\mathcal{R}_k$ is exactly scale-invariant. Let's sketch the proof in real-space and then write the result in Fourier space:

Once a mode is frozen outside the horizon, it may be written as

$$ds^2 = -dt^2 + a(t)^2 e^{2\mathcal{R}} d\mathbf{x}^2 \tag{3.29}$$

i.e. the long-wavelength curvature perturbation $\mathcal{R}_{\mathbf{k}_L}$ rescales the spatial coordinates within a given Hubble patch.

Therefore, the spectrum for the shorter wavelength modes $\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \rangle$ will depend on the value of the background fluctuations $\mathcal{R}_{\mathbf{k}_L}$ already frozen outside the horizon. Written in position space, the variation of the spectrum $\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \rangle$ given by the long-wavelength fluctuations $\mathcal{R}_L$ at linear order is given by

$$\frac{\partial}{\partial \mathcal{R}_L} \langle \mathcal{R}(x) \mathcal{R}(0) \rangle \cdot \mathcal{R}_L = x \frac{d}{dx} \langle \mathcal{R}(x) \mathcal{R}(0) \rangle \cdot \mathcal{R}_L \tag{3.30}$$

To find the three-point function, we then multiply eq. (3.30) by $\mathcal{R}_L$ and average over it. Written in Fourier space gives us eq. (3.26).

Again, this is what we would expect by our intuition: the long-wavelength moves frozen outside the horizon cannot have large interactions with the short-wavelength modes still within the horizon. Such modes well within the horizon oscillate, and any contributions to non-Gaussianities average out. This also provides an intuition as to when large non-Gaussianities *are* possible: Consider if all modes have similar wavelengths and exit the horizon at the same time. Large interactions will occur and therefore large non-Gaussian contributions will result. As we will discuss in the next section, such inflationary models which provide such interaction terms are due to higher derivative kinetic terms.

3.4 Large Non-Gaussianity from Inflation

We've seen that non-Gaussianity is always very small for all single-field slow-roll models. It was shown that under no other assumptions except single-field inflation (i.e. no other fields evolve during inflation, only a single inflaton) that such models produce negligible non-Gaussianity in the squeezed-limit.

Non-Gaussianities peaked at other signals may therefore arise in violations of our simplest vanilla inflationary model, i.e.

- single-field scalar inflation
- canonical kinetic term

- always slow-roll
- started in Bunch-Davis vacuum
- within Einsteinian gravity

For instance, single-field models can still give large non-Gaussianity if higher-derivative terms are important during inflation (i.e. as opposed to a canonical kinetic term and no higher-derivative corrections in slow-roll inflation).

Consider the action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}[R - P(X,\phi)], \quad \text{where} \quad X \equiv (\partial_\mu\phi)^2 \tag{3.31}$$

where $P(X < \phi)$ denotes arbitrary function of the kinetic term $X = (\partial_\mu\phi)^2$ and hence contains higher-derivative interactions. Such models may have a non-trivial speed of sound for the propagation of fluctuations:

$$c_s^2 \equiv \frac{P_{,X}}{P_{,X} + 2XP_{,XX}} \tag{3.32}$$

Analogously with our calculation in Appendix C, the second-order action for $\mathcal{R}$ (which recall gives the spectrum $P_{\mathcal{R}}$) is given by [100]

$$S_{(2)} = \int d^4x\epsilon\big[a^3(\dot{\mathcal{R}})^2/c_s^2 - a(\partial_i\mathcal{R})^2\big] + \mathcal{O}(\epsilon^2) \tag{3.33}$$

and the third-order action for $\mathcal{R}$ (giving the bispectrum $B_{\mathcal{R}}$) is given by

$$S_{(3)} = \int d^4x\epsilon^2\big[\ldots a^3(\dot{\mathcal{R}})^2/c_s^2 + \ldots a(\partial_i\mathcal{R})^2\mathcal{R} + \ldots a^3(\dot{\mathcal{R}})^3/c_s^2\big] + \mathcal{O}(\epsilon^3) \tag{3.34}$$

Note that the third-order action $S_{(3)}$ is suppressed by an extra factor of ϵ relative to the second-order action, which is a reflection of the fact that non-Gaussianity is small in the slow-roll limit, i.e. $P(X,\phi) = X - V(\phi)$, $c_s^2 = 1$. However, for small speeds of sound $c_s^2 \ll 1$ away from the slow-roll limit, a few interactions terms enlarge and non-Gaussianity can become significant. (We will discuss this in full next chapter).

The signal is peaked at the equilateral triangle configuration, with

$$f_{\mathrm{NL}}^{\mathrm{equil}} = -\frac{35}{108}\Big(\frac{1}{c_s^2} - 1\Big) + \frac{5}{81}\Big(\frac{1}{c_s^2} - 1 - 2\Lambda\Big) \tag{3.35}$$

where

$$\Lambda \equiv \frac{X^2 P_{,XX} + \frac{2}{3} X^3 P_{,XXX}}{X P_{,X} + 2X^2 P_{,XX}} \tag{3.36}$$

Whether actions with arbitrary $P(X, \phi)$ are consistent in high-energies theories is an important theoretical challenge for these models.

(Additionally, we note that most New Ekpyrotic models are predicted to also manifest sizable non-Gaussianity peaked at the squeezed triangle. Such a signal provides a way to falsify this alternative to inflation or favor it over other inflationary models.)

This explains the large non-Gaussianity peaked at equilateral triangles from inflationary models with higher-derivative terms (e.g. DBI inflation). What about the other models? Well, in contrast with single-field slow-roll, where the interactions of the inflaton are constrained via the slow-roll parameters dictating how inflation must occur, models have been written to circumvent this requirement. Such models (e.g. curvaton, inhomogeneous heating) create fluctuations via a second field which is not the inflaton (e.g. the "curvaton"). Such models give a non-Gaussian signal peaked at squeezed triangles.

Non-Gaussianities also arise if inflation started in a state other than the Bunch-Davies vacuum. It has been shown that non-Gaussianity may be detectable if inflation began in an excited state, with a signal such that

$$S^{\text{folded}}(k_1, k_2, k_3) \propto \frac{1}{K_{111}}(K_{12} - K_3) + 4\frac{K_2}{(\tilde{k}_1 \tilde{k}_2 \tilde{k}_3)^2} \tag{3.37}$$

The effect however is exponentially diluted when inflation lasts much longer than the minimal amount of $e-$foldings. To sketch one such example, consider that the Bunch-Davis vacuum normally has the positive energy mode $\sim e^{-ik\tau}$. If we simply add a component of a negative energy mode $\sim e^{+ik\tau}$, this would affect our momentum of our three-point function such that one k_i becomes $-k_i$, thereby resulting in an enhanced non-Gaussianity in the folded triangle limit. See [47, 101-104] for details.

One may question how unique such multifield models are in predicting unique non-Gaussian shapes. Differentiating between such signals either would clues from other sources (e.g. a significant detection of gravitational waves would rule against New Ekpyrotic models; a detection of isocurvature perturbations would rule in favor of curvaton models; etc.). As these models individually do rely upon different mechanisms, more work must be done to distinguish them on the basis of non-Gaussianity.

This is not to say however significant progress is not being made on this front:

To provide some intuition with regards to the complexity of extracting predictions of non-Gaussianity for multifield models, let's begin by considering extensions to single-field inflation via adding more scalar fields: we consider light fields of the order $\sim H$, as heavier fields $m \ggg H$ would not affect quantum fluctuations. Referring to these extra scalar fields as "isocurvatons", let's consider the case of one massless inflaton with massive isocurvatons:

Such massive isocurvatons decays quickly after exiting horizon via expansion. The rate at which this decay occurs is related to the isocurvaton mass: if $m > \sqrt{s}H$, then it decays faster, and if $m > \sqrt{s}H$, it decays slower (possibly long-after superhorizon scales).

The former case results in a signal peaking in the equilateral limit, thus corresponding to bispectra of quasi-equilateral shapes. The latter case results in a signal peaked in the squeezed limit $k_3 \lll k_1 = k_2$ such that the bispectrum shapes correlate to $(k_3/k_1)^{1/2-\nu}$ where ν goes from 0 to 3/2 (which corresponds to m equals $3H/2$ to 0).

In addition, let's consider these isocurvatons to be massless, such that the amplitudes of the isocurvaton fluctuations do not decay after exiting the horizon. In this case, such superhorizon modes may be studied classically and their evolution is therefore local in space (as they evolve independently form one another). Non-Gaussianities in this case are generated via the non-linearity of the isocurvaton evolution, resulting in a local shape bispectrum $f_{\rm NL}^{\rm local}$ peaked in the squeeze limit, given by the form

$$f_{\rm NL}^{\rm local} = \frac{5}{6}\frac{K_{pq}K_pK_q}{(K_r^2)^2} \tag{3.38}$$

This example demonstrates how complicated extracting predictions from multifield models can be.

We should also note recently proposed models [48, 49, 105-108] in the limit of superluminal sound speed $c_s \ggg 1$, partially motivating by earlier models of a decaying sound speed [66,109-111]. Such models give rise to highly scale-dependent non-Gaussianities. Such theories are not UV completed at $c_s \ggg 1$ using the S-matrix methods we've employed [112]. It can be shown [48,113] in such models that the Hubble parameter at freeze-out and the amplitude of the power spectrum are related by

$$H^2 = M_{\rm Pl}^2\frac{8\pi^2\epsilon}{(1+\epsilon)^2}c_sP_\zeta \tag{3.39}$$

Such an expression allows us to deduce constraints on such models, as in the $c_s \ggg 1$ limit, where the observed normalization $P_\zeta \sim 10^{-10}$ must match $H < M_{\rm Pl}$. Such a constraint implies a bound of approximately 10^{28} $e-$foldings of nearly scale-invariant and Gaussian perturbations.

In the phenomenological case previously described in Chapter 2 of the inflationary step-potential, i.e.

$$V(\phi) = \frac{1}{2}m^2\phi^2\left[1 + c\,\tanh\Big(\frac{\phi - b}{d}\Big)\right] \tag{3.40}$$

such a sharp feature in the potential results in a sharp change of slow-roll parameters, which may enhance the magnitudes of time-derivatives in many models. Thus would boost non-Gaussianities i modes near the horizon-exit. Such features may not be sharply peaked, but are recognizable via their periodicity, i.e. an oscillatory component in the interaction couplings resulting in oscillations before exiting the horizon [114]. Taking $k_* = -1\tau_*$ as the momentum of the mode near horizon-exit, such bispectra contain a sinusoidal factor $\sim \sin(K/k_*)$.

3.5 Current Constraints

The latest constraint on $f_{NL}^{\rm local}$ and $f_{NL}^{\rm equil}$ computed by Smith, Senatore, and Zaldarriaga [115,116] are given by

$$-4 \;<\; f_{\rm NL}^{\rm local} \;<\; +80 \quad \text{at } 95\% \qquad \text{CL} \tag{3.41}$$

$$-125 \;<\; f_{\rm NL}^{\rm equil} \;<\; +435 \quad \text{at } 95\% \qquad \text{CL} \tag{3.42}$$

Future experiments (e.g. the Planck satellite and the proposed CMBPol mission) are projected to measure at a sensitivity $\sigma(f_{\rm NL}^{\rm local}) \sim 5$ and $\sigma(f_{\rm NL}^{\rm local}) \sim 2$ respectively.

As previously detailed with respect to slow-roll and non-linear effects from CMB evolution, we currently expect to see a signal from secondary effects not associated with inflation at a sensitivity of at most $f_{\rm NL} \sim \mathcal{O}(1)$. We therefore must compute in detail how the non-linear evolution of fluctuations can induce its own non-Gaussianity; otherwise, we risk confusing these effects with the primordial signal. Although their order of magnitude is estimated in [117-119], these effects have not been precisely computed.

Clearly, a systematic and thorough account of all effects inducing observable levels of non-Gaussianity must be soon accomplished for this approach to be successful.

Chapter 4

The Effective Field Theory of Inflation

4.1 Motivating EFT Inflation

In the previous sections, we began with theoretical considerations of what sort of inflationary expansion is possible. We have examined various motivations for such models, reflecting on how consistent and "natural" such scenarios would be on the basis of a number of criteria. In addition, we have discussed the opposite approach of constructing inflationary theories inspired by possible observational anomalies (i.e. the inflationary step potential).

At this point, we may start to worry how systematic this entire process is. With the countless array of inflationary scenarios on the market, how could we possibly pick the "most natural" one, even with previously discussed observational constraints? It is highly unlikely that we could ever somehow measure the inflaton with particle accelerators. The prospect of confidently concluding whether inflation did or did not actually happen gradually begins to feel like a fantasy.

In this section, we introduce a more standardized approach which describes single-field inflation solely in terms relevant to observations. Not only does such an approach unify (virtually) all single-field inflationary models, it allows us to consistently explore its dynamics and associated observational predictions. The following procedure relies upon the techniques provided by effective field theory:

The key insight offered by effective field theory over the past several decades has been a simple but powerful one: physics at a particular energy scale, time scale, or distance scale will not depend sensitively on detailed knowledge of physics at widely different scales (see [120,121] for background). With this organizational principle, we are able to precisely isolate high-energy degrees of freedom from the low-energy degrees of freedom of the system, while systematically keeping track of their influence of the low-energy domain. These methods have been one of the most powerful tools of theoretical physics, indelibly affecting condensed matter to high-energy particle physics. Recently, these ideas have been applied to the realm of cosmology (see [122-124] for examples). Here, we offer an overview of a promising approach of this paradigm, the "Effective Field Theory of Inflation", originally developed [96].

Before proceeding to motivate and detail this theory in full, let's rather summarize the results of the EFT of single-clock inflation and sketch the details of how the formalism works, highlighting why such a theory is relevant. This short, cursory outline will serve as a reference point throughout the later discussion when we construct the theory in detail.

4.2 Outline of EFT Inflation

Recall our definition of inflation: it is a period of accelerated cosmic expansion with an approximately constant Hubble parameter H, i.e. $\dot{H} \ll H^2$. We know that the period of inflation must end to give way to the standard FLRW cosmology. This simple, but non-trivial insight implies that there is a physical "clock" of inflation which controls when inflation ends. Furthermore, perhaps the most appealing aspect of inflation is as the originator of the seeds of large scale structure, the initial fluctuations leading to the clusters of galaxies we see today. Indeed, not only does accelerated expansion end to way to a FLRW universe, but inflation must end at slightly different times in different regions of the universe, giving rise to the initial seeds of structure formation. The quantum fluctuations in the physical clock which determines when inflation ends are the source of these initial seeds.

Due to this physical clock that controls when inflation ends, time-translations of the quasi-de Sitter background (i.e. $|\dot{H}|/H^2 \ll 1$) are spontaneously broken and therefore the inflationary perturbations are the associated Goldstone bosons, denoted by π. These Goldstone modes are associated to the curvature perturbations in the scalar field $\delta\phi$, which at linear order is of the form $\pi = \delta\phi/\dot{\phi}$, such that $\dot{\phi}$ is the speed of the background solution. Note that contrary to our previously discussed methods of inflationary model building, the Goldstone mode π does not in fact assume any presence of a scalar field. Indeed, this description is far more general, as we are only assuming there is one relevant degree of freedom during the inflationary phase. Because the EFT description is general and valid for any background $H(t)$ which spontaneously breaks time-translation invariance, we may ignore the microscopic details specific to the theory which give rise to the de Sitter background.

We will later observe that this nuance allows us to consider inflationary models without explicitly postulating the physics of the theory. As opposed to arguing about whether some inflationary Lagrangian is the most "natural" and tinkering the consequent potential observables, the EFT approach does not actually have to explicitly discuss the various scalar fields, potentials, etc.—we only need to specify a few numbers via an action of perturbations to calculate how large the quantum fluctuations are! Such a non-trivial insight is a major motivation for this formalism.

At leading leading order we may relate Goldstone boson π to the standard curvature perturbation ζ such that $\zeta = -H\pi$. (Note that such a result is valid at linear order and also at leading order in the generalized slow-roll parameters). As usual, such curvature perturbations may lead to CMB anisotropies. The action written for these perturbations (via Goldstone modes) is highly constrained by symmetry (the standard result of Goldstone bosons). In fact, we will find that such non-linearly-realized time-translation symmetries of the quasi-de Sitter background constrain the sound speed c_s to large interactions, resulting in large equilateral non-Gaussianities, i.e $f_{\rm NL}^{\rm equil.} \sim c_s^{-2}$.

Assuming the Goldstone boson π is protected by an approximate shift symmetry which allows us to neglect terms such that π appears without a derivative acting on it, we find the most general Lagrangian for the Goldstone boson π is of the form

$$\begin{aligned} S_\pi = \int d^4x\sqrt{-g}\Bigg[&- \frac{M_{\rm Pl}^2\dot{H}}{c_s^2}\left(\dot{\pi}^2 - c_s^2\frac{1}{a^2}(\partial_i\pi)^2\right) \\ &+ \frac{\dot{H}M_{\rm Pl}^2}{c_s^2}(1-c_s^2)\dot{\pi}\frac{1}{a^2}(\partial_i\pi)^2 - \frac{\dot{H}M_{\rm Pl}^2}{c_s^2}(1-c_s^2)\left(1+\frac{2}{3}\frac{\tilde{c}_3}{c_s^2}\right)\dot{\pi}^3 \\ &- \frac{d_1}{4}HM^3\left(6\dot{\pi}^2 + \frac{1}{a^2}(\partial_i\pi)^2\right) - \frac{(d_2+d_3)}{2}M^2\frac{1}{a^4}(\partial_i^2\pi)^2 - \frac{1}{4}d_1M^3\frac{1}{a^4}(\partial_j^2\pi)(\partial_i\pi)^2 \\ &+ \dots \Bigg] \end{aligned} \tag{4.1}$$

where H is the standard Hubble parameter, and c_s is the speed of sound of the fluctuations (which since the background is not Lorentz invariant does not need to be equal to one). M is a free parameter with dimension of mass while d_1, d_s, d_3 and $\tilde{c}_3$ are dimensionless parameters expected to be of order one. Higher derivative terms (which generally give negligible contribution to the observables) are denoted by dots.

Observe the dynamics of this Lagrangian. In the de Sitter limit $\dot{H} \to 0$, the leading kinetic term $\dot{H} M_{\rm Pl}^2 (\partial_i \pi)^2$ goes to zero and terms in the last line (which relate to the extrinsic curvature of the system) become relevant. Otherwise, away from this limit we see this Lagrangian contains only two leading interaction terms, $\dot{\pi}(\partial_i \pi)^2$ and $\dot{\pi}^3$. Moreover, the coefficients are not constrained by the de Sitter space limit and may therefore induce detectable non-Gaussianities. For instance, observe the coefficient of $\dot{\pi}(\partial_i \pi)^2$ is correlated with the sound speed of the fluctuations. (This also holds for the coefficient of $\dot{\pi}^3$, which is also dependent on the parameter $\tilde{c}_3$, which is expected to be 1 due to UV completion; see details in [19]). This result justifies our previously claim that large non-Gaussianities are correlated to small sound speeds c_s, the velocities at which fluctuations propagate. Consider this as another major advantage of the EFT formalism: unifying all single-field models allows us to explore the signature space of single-field inflation in full generality, thus discerning such correlations via each individual operator.

4.3 Single-Field EFT Inflation

We will now proceed by attentively motivating and detailing the Effective Field Theory of Inflation before explicitly constructing the action.

Observe that in inflation, time-diffeomorphisms are broken and therefore there exists a Goldstone boson associated with it symmetry breaking. In practice, we choose a particular time-slicing such that the clock field is taken to be uniform. Because inflation spontaneously breaks time-translation symmetry, we can construct an effective action of the Goldstone boson associated with this spontaneous symmetry breaking. Later, we will detail how this theory is highly constrained by non-linearly realized symmetries of the quasi-de Sitter background.

The most general effective action is then constructed by writing down all operators that are functions of the metric fluctuations and invariant under time-dependent spatial diffeomorphisms. The two most important objects appearing in this construction are the metric perturbation $\delta g^{00} = g^{00} + 1$ and the extrinsic curvature perturbation $\delta K_{\mu\nu} = K_{\mu\nu} - a^2 H h_{\mu\nu}$, where $h_{\mu\nu}$ is the induced metric on the spatial slices. We use these geometrical quantities to write down the most general action with unbroken spatial diffeomorphisms, given by the form

$$S = \int d^4x\sqrt{-g}\,\Big[\,\frac{1}{2}M_{\rm Pl}^2 R + M_{\rm Pl}^2 \dot{H} g^{00} - M_{\rm Pl}^2(3H^2 + \dot{H}) + \frac{1}{2!}M_2^4(t)(\delta g^{00})^2 + \frac{1}{3!}M_3^4(t)(\delta g^{00})^3 + \ldots - \frac{1}{2}\bar{M}_1^3(t)\delta g^{00}\delta K^\mu_\mu - \frac{1}{2}\bar{M}_2^2(t)(\delta K^\mu_\mu)^2 - \frac{1}{2}\bar{M}_3^2(t)\delta K^{\mu\nu}\delta K_{\mu\nu} + \ldots\Big] \tag{4.2}$$

Before continuing to detail the terms of this action, we should first derive it.

4.3.1 Deriving the Action in the Unitary Gauge

Let's begin with constructing the action for the single-field (single-clock) case.

As we've previously mentioned, the effective field theory approach has been used very successfully to high-energy particle physics to condensed matter systems. The general approach is to describe a system through the lowest dimension operators which are compatible with the underlying symmetries. Usually when constructing single-field inflationary models, one presents a Lagrangian for a scalar field (the so-called "inflaton" ϕ responding for inflation) and solve the equation of motion for ϕ together with the Friedman equation for the FLRW metric (as we first accomplished in Chapter 2).

In contrast, we will now begin by taking an inflating solution, i.e. an accelerated expansion with a slowly varying Hubble parameter H with the scalar following a homogeneous time-dependent solution $\phi_0(t)$. Using perturbation theory, we then split the inflaton ϕ into an unperturbed part (i.e. the background) plus a fluctuating one, such that

$$\phi(\vec{x}, t) = \phi_0(t) + \delta\phi(\vec{x}, t) \tag{4.3}$$

where we note that while ϕ is a scalar under all diffeomorphisms, the perturbation $\delta\phi$ is a scalar only under spatial diffeomorphisms while it transforms non-linearly with respect to time diffeomorphisms, such that

$$t \to t + \xi^0(t, \vec{x}), \qquad \delta\phi \to \delta\phi + \dot{\phi}_0(t)\xi^0 \tag{4.4}$$

Let's now apply the unitary (or comoving) gauge for which $\delta\phi = 0$ i.e. $\phi(\vec{x}, t) = \phi_0(t)$. In this gauge, there are no inflaton perturbations, but all degrees of freedom are in the metric. There are no matter fluctuations, only metric fluctuations.

The scalar variable $\delta\phi$ is hidden in the metric or equivalently has been "eaten" by the graviton, which now has three degrees of freedom: the scalar mode and the two tensor helicities. This scenario is analogous to a spontaneously broken gauge theory where a Goldstone mode, which transforms non-linearly under the gauge symmetry, is eaten by the gauge boson (i.e. the unitary gauge) to give rise to a massive spin-1 particle. Also, we observe that once this is implemented, our Lagrangian is no longer invariant under full space-time diffeomorphisms but only under spatial reparametrizations.

We will now exploit the implication that inflation spontaneously breaks time-translation symmetry. In inflation, time-diffeomorphisms are broken and therefore there exists a Goldstone boson associated with it symmetry breaking. In practice, we chose a particular time-slicing such that the clock field is taken to be uniform. We the can construct the most general Lagrangian with the lowest dimension operators invariant under spatial diffeomorphisms.

We begin by writing down operators that are functions of the metric $g_{\mu\nu}$ and invariant under the time-dependent spatial diffeomorphisms $x^i \to x^i + \xi^i(t, \vec{x})$. Note that spatial diffeomorphisms remain unbroken.

Let's first write down the most general action invariant under spatial diffeomorphisms in the unitary gauge. We use a preferred time slicing with function $\tilde{t}(x)$, which non-linearly recognizes time diffeomorphisms (e.g. if the breaking is given by a time-evolving scalar, surfaces of constant $\tilde{t}$ are also of constant value of the scalar). The unitary gauge is chosen such that the time coordinate t coincides with $\tilde{t}$, so the additional degree of freedom $\tilde{t}$ does not explicitly appear in the action.

The most general space diffeomorphism-invariant action in the unitary gauge can be written as

$$S = \int d^4\sqrt{-g}F(R_{\mu\nu\rho\sigma}, g^{00}, K_{\mu\nu}, \nabla_\mu, t) \tag{4.5}$$

where all free indices in the function F must be upper 0's. Let's motivate this result by considering the various terms we can build:

- Terms invariant under all diffeomorphisms
- Any generic function $f(\tilde{t})$ becomes $f(t)$ in the unitary gauge, and so we may use any generic functions of time in front of any terms in the Lagrangian.
- Polynomials of the Riemann tensor $R_{\mu\nu\rho\sigma}$ and its covariant derivatives are invariant under all diffeomorphisms. Note that the metric and the completely antisymmetric tensor $(-g)^{-1/2\epsilon^{\mu\nu\rho\sigma}}$ are used to contract the indices to give scalars.

- The gradient $\partial_\mu \tilde{t}$ becomes δ^0_μ in the unitary gauge, so we can leave a free upper free index 0 in every tensor e.g. we can use g^{00} (and functions thereof) or the component of the Ricci tensor R^{00}.

- In order to define the induced spatial metric on surfaces of constant $\tilde{t}$, such that $h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$, we define a unit vector perpendicular to said surfaces:

$$n_\mu = \frac{\partial_\mu \tilde{t}}{\sqrt{-g^{\mu\nu}\partial_\mu \tilde{t}\partial_\nu \tilde{t}}} \tag{4.6}$$

  Hence every tensor can be project on said surfaces using $h_{\mu\nu}$. In particular, we may use in our action the Riemann tensor of induce 3D metric ${}^{(3)}R_{\alpha\beta\gamma\delta}$ and covariant derivatives with respect to this 3D metric.

- Likewise, we consider the covariant derivatives of $\partial_\mu \tilde{t}$. The covariant derivatives of n_μ is a derivative action on the normalization factor which gives terms like $\partial_\mu g^{00}$ which are covariant on their own, and therefore may be used in the unitary gauge action. Note the covariant derivative of n_μ projected on surface of constant $\tilde{t}$ gives the extrinsic curvature of these surfaces $K_{\mu\nu} \equiv h^\sigma_\mu \nabla_\sigma n_\nu$, where the index ν is already project on the surface because $n^\nu \nabla_\sigma n_\nu = \frac{1}{2}\nabla_\sigma(n^\nu n_\nu) = 0$. The covariant derivative of n_ν perpendicular to the surface can be rewritten as $n^\nu \nabla_\sigma n_\nu = -\frac{1}{2}(-g^{00})^{-1} h^\mu_\nu \partial_\mu(-g^{00})$ so as not to give rise to new terms. Therefore, we can write all covariant derivatives of n_μ can be written using the extrinsic curvature $K_{\mu\nu}$ (and its covariant derivatives) and derivatives of g^{00}.

- The Riemann tensor of the induced 3D metric and the extrinsic curvature is redundant as ${}^{(3)}R_{\alpha\beta\gamma\delta}$ can be rewritten using the Gauss-Codazzi relation as in Wald [125]

$${}^{(3)}R_{\alpha\beta\gamma\delta} = h^\mu_\alpha h^\nu_\beta h^\rho_\gamma h^\sigma_\delta R_{\mu\nu\rho\sigma} - K_{\alpha\gamma}K_{\beta\delta} + K_{\beta\gamma}K_{\alpha\delta} \tag{4.7}$$

Therefore, using the two most important objects i.e. the metric perturbation $\delta g^{00} = g^{00} + 1$ and the extrinsic curvature perturbation $\delta K_{\mu\nu} = K_{\mu\nu} - a^2 H h_{\mu\nu}$, where $h_{\mu\nu}$ is the induced metric on the spatial slices, the most general Lagrangian may be written as

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2 R - c(t)g^{00} - \Lambda(t) + \frac{1}{2!}M_2(t)^4(g^{00}+1)^2 + \frac{1}{3!}M_3(t)^4(g^{00}+1)^3 + \\ - \frac{\bar{M}_1(t)^3}{2}(g^{00}+1)\delta K^\mu{}_\mu - \frac{\bar{M}_2(t)^2}{2}(g^{00}+1)\delta K^\mu{}_\mu{}^2 - \frac{\bar{M}_3(t)^2}{2}(g^{00}+1)\delta K^\mu{}_\nu K^\nu{}_\mu + \dots\Big] \tag{4.8}$$

where the trailing dots denote terms of higher order in the fluctuations or with more derivatives. Observe that the first term is the Einstein-Hilbert term, and that the first three terms are the only terms which start linearly in the metric fluctuations around the chosen FLRW solution. All others are explicitly of quadratic order or higher, as the action must start in the quadratic order in the fluctuations.

We now want to write the Lagrangian as a polynomial of linear terms like $\delta K_{\mu\nu}$ and $g^{00}+1$ to make it evident whether an operator starts at linear, quadratic or higher orders. All linear terms shall contain derivatives and can be integrated by parts to give a combination of linear terms plus covariant terms of higher order. We begin by fixing the coefficients $c(t)$ and $\Lambda(t)$ by the requirement of having a given FLRW solution $H(t)$, which is equivalent to canceling all tadpole terms around this solution i.e. the terms proportional to c and Λ giving a stress-energy tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}} \tag{4.9}$$

Note that $T^{\mu\nu}$ does not vanish at zeroth order in the perturbations and therefore contributes to the right hand side of the Einstein equations (see Appendix B in [96]).

During inflation we are interested in a flat FLRW universe such that the Friedman equations are given by

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\big[c(t)+\Lambda(t)\big], \qquad \frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{3M_{\rm Pl}^2}\big[2c(t)-\Lambda(t)\big] \tag{4.10}$$

It can be shown that by considering fluctuations around a FLRW background (i.e.

solving for c and Λ) we find

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2 R + M_{pl}^2\dot{H}g^{00} - M_{\rm Pl}^2(3H^2+\dot{H}) + \\ + \frac{1}{2!}M_2^4(t)(\delta g^{00})^2 + \frac{1}{3!}M_3^4(t)(\delta g^{00})^3 + \dots \\ - \frac{1}{2}\bar{M}_1^3(t)\delta g^{00}\delta K_\mu^\mu - \frac{1}{2}\bar{M}_2^2(t)(\delta K_\mu^\mu)^2 - \frac{1}{2}\bar{M}_3^2(t)\delta K^{\mu\nu}\delta K_{\mu\nu} + \dots\Big] \tag{4.11}$$

which is the Lagrangian introduced here in eq. (4.2).

Again, recall that the first term is the Einstein-Hilbert term. Only terms which start linearly in the metric fluctuations, shown in the first line. The second line in eq. (4.11) contains terms which start quadratic in the fluctuations and have no derivatives. Higher derivative terms are shown in the third line. Thus, dots denote operators starting at a higher order in derivatives or in the perturbations.

This describes the most generic Lagrangian not only just for the scalar mode, but also for gravity, as higher effects will be encoded in corrections of higher derivative terms. It is in fact the most general action of single-field ("single-clock") inflation, and moreover, it is unique.

Observe that the operator coefficients $H(t), M_n(t)$, and $\bar{M}_n(t)$ are of a generic time-dependence; however we are interested in solutions such that $|\dot{H}| << H^2$ and therefore such time-dependence is negligible. The effective field theory approach allows us to organize the action in terms of a low-energy expansion of the system's fields and their derivatives. Because g^{00} is a scalar with zero derivatives acting on it, while $K_{\mu\nu}$ is a one-derivative object, in most situations the dynamics are dominated by the terms involving g^{00}.

Let's observe how the effective action eq. (4.11) unifies single-field models of inflation, a considerably advantage of this formalism:

- Note that the first line in eq. (4.11) characterizes all single-field slow-roll models of inflation. In a model with minimal kinetic term and a slow-roll potential $V(\phi)$, this may be written in the unitary gauge as:

$$\int d^4x\sqrt{-g}\Big[-\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\Big] \to \int d^4x\sqrt{-g}\Big[-\frac{1}{2}\dot{\phi}^2 g^{00} - V\big(\phi_0(t)\big)\Big] \tag{4.12}$$

where $-\frac{1}{2}\dot{\phi}(t)_0^2 = M_{\rm Pl}^2|\dot{H}|$ and $V\big(\phi(t)\big) = M_{\rm Pl}^2(3H^2+\dot{H})$ are given by the Friedman equations. In such a case all other terms encode possible effects of high energy physics on the simple slow-roll model.

- The second line in eq. (4.11) parameterizes all possible Lagrangian with non-trivial kinetic terms, i.e. with at most one derivative acting on each ϕ. Denoting $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and evaluating at $\phi_0(t)$

$$\mathcal{L} = P(X,\phi) \to P(\dot{\phi}_0(t)^2 g^{00}, \bar{\phi}) \implies M_n^4 = \dot{\bar{\phi}}^{2n}\frac{\partial^n P}{\partial \bar{X}^n} \tag{4.13}$$

The operators proportional to M_n^4 start at the $n-$order in the fluctuations. We will later discuss how the coefficient M_2 induces a sound speed in the quadratic action

$$c_s^{-2} \equiv 1 - \frac{2M_2^4}{M_{\rm Pl}^2\dot{H}} \tag{4.14}$$

thus deviating from the speed of light, $c_s \lesssim 1$.

- As we previously discussed, the last line in eq. (4.11) describes all terms with higher derivatives that cannot be eliminated by partial integrations, e.g. $(\Box\phi)^2$. Normally these terms become negligible via the threshold of the physics (i.e. usually extra powers of the cutoff suppress such terms). However, they can become important in cases like Ghost inflation [20,21] where the leading terms vanish because $M_{\rm Pl}^2\dot{H} \to 0$.

4.4 Reintroducing the Goldstone Boson

4.4.1 Review of Non-Abelian Gauge Theory

Before continuing, let's review some aspects of non-Abelian gauge theory, in particular how to reintroduce the Goldstone boson.

Consider a non-Abelian gauge theory with Lagrangian

$$\mathcal{L} = -\frac{1}{4}{\rm Tr}F_{\mu\nu}^2 - \frac{1}{2}m^2{\rm Tr}A_\mu^2 \tag{4.15}$$

where $A_\mu = A_\mu^a T^a$.

Under a gauge transformation this results in

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger \equiv \frac{i}{g} U D_\mu U^\dagger \tag{4.16}$$

with the action consequently becoming

$$\mathcal{L} = \frac{1}{4} \mathrm{Tr} F_{\mu\nu}^2 - \frac{1}{2} \frac{m^2}{g^2} \mathrm{Tr} D_\mu U^\dagger D^\mu U \tag{4.17}$$

Observe that we can then restore gauge invariance with the so-called Stückelberg trick: First we define $U = e^{i\pi^a T^a}$, where T^a is a generator of the group. By choosing unitary gauge, $\pi \equiv 0$ we reproduce the action in eq. (4.15). Including the Goldstone scalars π^a, the Lagrangian becomes gauge-invariant and can be expanded as

$$\mathcal{L} = -\frac{1}{4} \mathrm{Tr} F_{\mu\nu}^2 - \frac{1}{2} m^2 \mathrm{Tr} A_\mu^2 + \frac{1}{2} \frac{m^2}{g^2} (\partial_\mu \pi^a)^2 + i \frac{m^2}{g} \mathrm{Tr} \partial_\mu \pi^a T^a A^\mu + c.c. + \dots \tag{4.18}$$

To understand what this means, recall the "Goldstone Equivalence Theorem" from high-energy particle physics. First proved by Cornwall, Levin, Tiktopoulos, and Vayonakis [126], the Equivalence Theorem involves

> "[A] certain conservation of degrees of freedom. A massless gauge boson, which has two transverse polarization states, combines with a scalar Goldstone boson to produce a massive vector particle, which has three polarization states. When the massive vector particle is at rest, its three polarization states are completely equivalent, but when it is moving relativistically, there is a clear distinction between the transverse and longitudinal polarization directions. This suggests that a rapidly moving, longitudinally polarized massive gauge boson might betray its origin as a Goldstone boson." [127]

In this case, a Goldstone boson which transforms non-linearly under the gauge transformation provides the longitudinal component of a massive gauge boson. At sufficiently high energy such Goldstone boson becomes the only relevant degree of freedom. Hence, for a sufficiently high energy, the mixing with gravity becomes irrelevant and the scalar π becomes the only relevant mode in the dynamics. This is the so-called decoupling regime.

So, the purpose of reintroducing the Goldstones above in eq. (4.18) is to make the dynamics more transparent: for higher energies $E \gg m$, the scattering of Goldstone bosons describes well the scattering of the longitudinal mode of the gauge field. By taking the decoupling limit $m \to 0$ and $g \to 0$ with $m/g \equiv f_\pi$ fixed, the Goldstone bosons decouple from A_μ and leave us with

$$\mathcal{L} = -\frac{1}{2} f_\pi^2 \mathrm{Tr} \partial_\mu U^\dagger \partial^\mu U \tag{4.19}$$

Upon restoration of finite m and g, we expect corrections to the results from pure Goldstone boson scattering, perturbative in m/E and g^2, where E is the energy of the vector boson.

In consideration of what we accomplished above, let's now consider the case of broke time diffeomorphisms, concentrating on two operators:

$$\int d^4x \sqrt{-g} \Big[A(t) + B(t) g^{00}(x) \Big] \tag{4.20}$$

Under a broken time diffeomorphisms $t \to \tilde{t} + \xi_0(x)$, $\vec{x} \to \tilde{\vec{x}} = \vec{x}$, we note that g^{00} transforms as

$$g^{00}(x) \to \tilde{g}^{00}(\tilde{x}(x)) = \frac{\partial \tilde{x}^0(x)}{\partial x^\mu} \frac{\partial \tilde{x}^0(x)}{x^\nu} g^{\mu\nu}(x) \tag{4.21}$$

An therefore, under the transformed fields, the resulting action will be written as

$$\int d^4x \sqrt{-\tilde{g}(\tilde{x}(x))} \left| \frac{\partial \tilde{x}}{\partial x} \right| \left[A(t) + B(t) \frac{\partial x^0}{\partial \tilde{x}^\mu} \frac{\partial x^0}{\partial \tilde{x}^\nu} \tilde{g}^{\mu\nu}(\tilde{x}(x)) \right] \tag{4.22}$$

By changing integration variable to $\tilde{x}$, we find

$$\int d^4x \sqrt{-\tilde{g}(\tilde{x})} \Big[A(\tilde{t} - \xi^0(x(\tilde{x})) + B(\tilde{t} - \xi^0(x(\tilde{x})) \frac{\partial(\tilde{t} - \xi^0(x(\tilde{x})))}{\partial \tilde{x}^\mu} \frac{\partial(\tilde{t} - \xi^0(x(\tilde{x})))}{\partial \tilde{x}^\nu} \tilde{g}^{\mu\nu}(\tilde{x}) \Big] \tag{4.23}$$

Reintroducing the Goldstone boson is similar to the gauge theory scenario we reviewed previously. For every ζ^0 in the action above, we simply make the substitution $\xi^0(x(\tilde{x})) \to -\tilde{\pi}(\tilde{x})$, which results in

$$\int d^4x \sqrt{-g(x)} \Big[A(t + \pi(x)) + B(t + \pi(x)) \frac{\partial(t + \pi(x))}{\partial x^\mu} \frac{\partial(t + \pi(x))}{\partial x^\nu} g^{\mu\nu}(x) \Big] \tag{4.24}$$

where we've dropped the tildes for simplicity. Upon assigning to π the transformation rule $\pi(x) \to \tilde{\pi}(\tilde{x}(x)) = \pi(x) - \xi^0(x)$, we can further check that the action above is invariant under diffeomorphisms at all orders, not just for infinitesimal transformations.

Let's pause to consider what we've accomplished so far: notice the unitary Lagrangian eq. (4.11) describes three degrees of freedom: one scalar mode (the Goldstone mode π) and two graviton helicities. By the Equivalence Theorem, we know the by studying the scalar Goldstone mode at sufficiently high energies, we can observe the physics of the longitudinal components of massive gauge bosons. Similarly in EFT inflation, at higher energies the mixing with gravity becomes irrelevant and we can concentrate on the Goldstone scalar, where we can retrieve information about cosmological perturbations.

Using the above procedure on the unitary action eq. (4.11), we find the Lagrangian

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\Big[&\frac{1}{2}M_{\rm Pl}^2 R - M_{\rm Pl}^2\Big(3H^2(t+\pi) + \dot{H}(t+\pi)\Big) + \\
&+ M_{\rm Pl}^2\dot{H}(t+\pi)\Big((1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi g^{0i} + g^{ij}\partial_i\pi\partial_j\pi\Big) + \\
&\frac{M_2(t+\pi)^4}{2!}\Big((1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi g^{0i} + g^{ij}\partial_i\pi\partial_j\pi + 1\Big)^2 + \\
&\frac{M_3(t+\pi)^4}{3!}\Big((1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi g^{0i} + g^{ij}\partial_i\pi\partial_j\pi + 1\Big)^3 + \dots\Big]
\end{aligned}
\tag{4.25}
$$

By reintroducing the Goldstone π from the unitary gauge Lagrangian, we can study the physics of the Goldstone mode π at very short distances neglecting metric fluctuations. We see above that the quadratic terms which mix π and $g_{\mu\nu}$ contain fewer derivatives than the kinetic term of π so that they can be neglected above some high energy scale, which depends on which operators are present. In the simplest case where only the tadpole terms are relevant, i.e. $(M_2 = M_3 = \ldots = 0)$, corresponds to the standard slow-roll inflation cases where the leading term mixing with gravity comes from a term of the form

$$\sim M_{\rm Pl}^2\dot{H}\dot{\pi}\delta g^{00} \tag{4.26}$$

After canonical normalization such that $\pi_x \sim M_{pl}\dot{H}^{1/2}\pi$ and $\delta g_c^{00} \sim M_{pl}\delta g^{00}$, we find that the mixing terms can be neglected for energies above $E_{\rm mix} \sim \epsilon^{1/2}H$ (where ϵ denotes usual slow-roll parameter).

Consider another case in which the operator M_2 becomes large. Here will we find mixing terms of the form

$$\sim M_2^4\dot{\pi}\delta g^{00} \tag{4.27}$$

which similarly, upon canonical normalization $\pi_c \sim M_2^2\pi$, becomes negligible at energies larger than $E_{\rm mix} \sim M_2^2/M_{\rm Pl}$

So, to concentrate on the effects that are not dominated by mixing with gravity, we must neglect the metric perturbations and just keep the π fluctuation. Here, a term of the form g^{00} in the unitary gauge Lagrangian becomes

$$g^{00} \to -1 - 2\dot{\pi} - \dot{\pi}^2 + \frac{1}{a^2}(\partial_i\pi)^2 \tag{4.28}$$

Let's now assume that π has an approximate continuous shift symmetry which becomes exact when spacetime is precisely de Sitter. This allows us to neglect terms in π without a derivative that are generated by the time dependence of the coefficients in the action eq. (4.25). Therefore, in the regime $E \gg E_{\rm mix}$, the action eq. (4.25) becomes

$$S_\pi = \int d^2x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2R - M_{\rm Pl}^2\dot{H}\left(\dot{\pi}^2 - \frac{(\partial_i\pi)^2}{a^2}\right) + 2M_2^4\left(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\right) - \frac{4}{3}M_3^4\dot{\pi}^3 + \dots\right] \tag{4.29}$$

where ... denote terms we've ignored which come from the extrinsic curvature, as they are usually important only in the de Sitter limit, $\dot{H} \to 0$. Notice that this Lagrangian is of the same form as eq. (4.1). As usual with EFT methods, the details of UV effects of new physics are parametrized in higher dimensional operators.

Note that away from the de Sitter limit and $M_2 \neq 0$, we find the speed of sound to not equal to 1, $c_s \neq 1$, by the relation

$$M_2^4 = -\frac{1-c_s^2}{c_s^2}\frac{M_{\rm Pl}^2\dot{H}}{2} \tag{4.30}$$

At this order we have two independent cubic self-interactions $\dot{\pi}(\partial_i\pi)^2$ and $\dot{\pi}^3$ which can induce detectable non-Gaussianities in the primordial density perturbations. A large M_2 forces large self-interactions of the form $\dot{\pi}(\partial_i\pi)^2$ while the $\dot{\pi}^3$ term is not fixed as it also depends on M_3. We will later discuss how to constrain these parameters with cosmological data, a process similar to Precision Electroweak Tests of particle accelerators [128,129].

Although our goal is to compute predictions for future observations based on a given inflationary model, one may be concerned that decoupling limit S_π in eq. (4.29) would be irrelevant for these extremely infrared scales. However, recall that in the case of simple slow-roll single field inflation, the usual curvature perturbation ζ is constant at superhorizon scales at any order in perturbation theory, i.e. $\dot{\zeta} \to 0$ [130-131] and that the coefficients of the action are constrained by slow-roll parameters. In this case, we can think of this as calculating correlation functions just after the horizon crossing, which is valid for all observables not dominated by mixing with gravity.

However, we are still able to calculate the tilt of the spectrum with S_π in eq. (4.29). Therefore, by setting limits on higher order terms from such constraints, we can calculate all predictions of this model with the action eq. (4.11), even when the mixing with gravity is considerable.

(For more discussion on the relation between low-energy EFT physics and UV-completion for this theory, see [132]).

4.5 Non-Gaussianities from EFT Single-Field Inflation

4.5.1 The Slow-Roll Scenario

Consider the slow-roll case discussed in Chapter 2. We will begin by recalculating the main results in this formalism:

Start by setting all higher order operators to zero, i.e. $M_2 = M_3 = \bar{M}_1 = \bar{M}_2 \ldots = 0$. As justified previously, we'll concentrate on making predictions at scale H with the Goldstone Lagrangian eq. (4.29), neglecting mixing with gravity. We will calculate the conserved ζ soon after horizon crossing. By choosing the unitary gauge $\pi = 0$, we may write the spatial metric at linear order in the form

$$g_{ij} = a^2(t)\big[\big(1 + 2\zeta(t,\vec{x})\big)\delta_{ij} + \gamma_{ij}\big] \tag{4.31}$$

where γ is transverse and traceless, and describes two tensor degrees of freedom, and $\zeta(t,\vec{x}) = -H\pi(t,\vec{x})$.

Recall that for each mode k, we are only interested in the dynamics around horizon crossing $\omega(k) = k/a \sim H$. Note that since the background may be approximated as de Sitter up to slow-roll corrections during this period, the two-point function is therefore

$$\langle \pi_c(\vec{k}_1)\pi_c(\vec{k}_2)\rangle = (2\pi)^3\delta(\vec{k}_1 + \vec{k}_2)\frac{H_*^2}{2k_1^3} \tag{4.32}$$

where π_c denotes the canonically normalized scalar and $*$ denotes the value of a quantity at horizon crossing. Via our relation $\zeta = -H\pi$, we hence find the spectrum of ζ is given by

$$\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\rangle = (2\pi)^3\delta(\vec{k}_1+\vec{k}_2)\frac{H_*^4}{4M_{\rm Pl}^2|\dot{H}_*|}\frac{1}{k_1^3} = (2\pi)^3\delta(\vec{k}_1+\vec{k}_2)\frac{H_*^2}{4\epsilon_* M_{\rm Pl}^2}\frac{1}{k_1^3} \tag{4.33}$$

This is exact for all k up to slow-roll corrections since ζ is constant outside the horizon, which allows us to calculate the tilt of the spectrum at leading order in the slow-roll case

$$n_s - 1 = \frac{d}{d\log k}\log\frac{H_*^4}{|\dot{H}_*|} = \frac{1}{H_*}\frac{d}{dt_*}\log\frac{H_*^4}{|\dot{H}_*|} = 4\frac{\dot{H}_*}{H_*^2} - \frac{\ddot{H}_*}{H_*\dot{H}_*} \tag{4.34}$$

Note that when the leading result comes from mixing with gravity, the Lagrangian eq. (4.29) cannot allow us to calculate all observables and we must examine subleading corrections, i.e. re-examine our choice to set all higher order terms to zero. In addition, observe that arbitrarily setting higher-order terms to zero is obviously not very rigorous. Instead, we can set experimental limits on these operators, which is the very principle and advantage of the EFT approach! However, instead of relying upon detailed results of particle colliders (as we confined by more restricted observations from cosmological surveys), we must extrapolate such limits with of constraints slow-roll parameters via the primordial power spectra detailed in Chapter 2. For instance, limits on the tensor spectral index $n_t = -2\epsilon_*$ would constrain the higher-term operators and provide insight into the scale as to where such physics is relevant.

4.5.2 Large Non-Gaussianities and the Sound Speed

Note that because our background is not Lorentz invariant, the sound speed of the fluctuations c_2 is not required to equal 1.

Consider the coefficient of the time kinetic term $\dot{\pi}^2$ in eq. (4.29), which is not fully fixed by the background evolution

$$(-M_{\rm Pl}^2\dot{H} + 2M_2^4)\dot{\pi}^2 \tag{4.35}$$

In order to avoid system instabilities and an FLRW universe with $\dot{H} > 0$ (violating the null energy condition [133-134]), the speed of sound is written as

$$c_s^{-2} = 1 - \frac{2M_2^4}{M_{\rm Pl}^2\dot{H}} \tag{4.36}$$

Thus, to avoid superluminal propagation, which implies the theory has no Lorentz-invariant UV completion [112], we must have $M_2^4 > 0$. For more details on superluminal models, see Appendix E.

Using this definition and $\omega^2 = c_s^2 k^2$, we assume that the Goldstone boson is protected by an approximate shift symmetry allowing us to neglects terms where π appears without a derivative acting on it. We therefore write the most general Goldstone boson action as eq. (4.1), where note mixing with gravity may be neglected at energies $E \gg E_{\rm mix} \simeq M_2^2/M_{\rm Pl}$, implying predictions for cosmological observables are possible at $H \gg M_2^2/M_{\rm Pl}$.

Analogously to the previous section, we calculate the two-point function to be

$$\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\rangle = (2\pi)^3\delta(\vec{k}_1+\vec{k}_2)\frac{1}{c_{s*}}\cdot\frac{H_*^4}{4M_{\rm Pl}^2|\dot{H}_*|}\frac{1}{k_1^3} = (2\pi)^3\delta(\vec{k}_1+\vec{k}_s)\frac{1}{c_{s*}}\cdot\frac{H_*^2}{4\epsilon_* M_{\rm Pl}^2}\frac{1}{k_1^3} \quad (4.37)$$

with contributions to the tilt due to the variation with time of the sound speed, such that

$$n_s = \frac{d}{d\log k}\log\frac{H_*^4}{|\dot{H}_*|c_{s*}} = \frac{1}{H_*}\frac{d}{dt_*}\log\frac{H_*^4}{|\dot{H}_*|c_{s*}} = 4\frac{\dot{H}_*}{H_*^2} - \frac{\ddot{H}_*}{\dot{H}_* H_*} - \frac{\dot{c}_{s*}}{c_{s*}H_*} \quad (4.38)$$

Returning to the action S_π in eq. (4.29), observe that the coefficient of operators $\dot{\pi}(\partial_i\pi)^2$ and $\dot{\pi}^3$, which implies a reduced speed of sound enhances the three-point correlator and therefore non-Gaussianities. Using the EFT approach, it becomes evident that the coefficient of $\dot{\pi}(\partial_i\pi)^2$ gives the leading order non-Gaussian contribution in the limit of a small sound speed $c_s \ll 1$ (i.e. spatial derivatives are enhanced with respect to time derivatives as the mode freezes with $k \sim H/c_s$), a signature predicted in several inflationary models (e.g. DBI inflation).

In the limit $c_s \ll 1$, the quartic terms in the Lagrangian freeze (at around $\omega \sim H$) which leaves the operator $\dot{\pi}(\partial_i\pi)^2$ as the leading term. We can find the associated level of non-Gaussianity to be given by the ratio

$$\frac{\mathcal{L}_{\dot{\pi}(\nabla\pi)^2}}{\mathcal{L}_2} \sim \frac{H\pi\left(\frac{H}{c_s*}\pi\right)^2}{H^2\pi^2} \sim \frac{H}{c_s^2} \sim \frac{1}{c_s^2}\zeta \quad (4.39)$$

which allows us to deduce the magnitude of non-Gaussianities in terms of parameters $f_{\rm NL}$. Note that we used the linear order relation $\zeta = -H\pi$ where $\zeta \sim 10^{-5}$ can be used to estimate the non-linear corrections. Parametrizing $f_{\rm NL}$ such that $\mathcal{L}_{\dot{\pi}(\partial_i\pi)^2}/\mathcal{L}_2 \sim$

$f_{\rm NL}\zeta$, we find the leading contribution to be

$$f^{\rm equil}_{{\rm NL},\dot{\pi}(\partial_i\pi)^2} \sim \frac{1}{c_s^2} \tag{4.40}$$

a clear negative correlation between non-Gaussianity and sound speed.

Considering the additional contribution to the three-point function $-\frac{4}{3}M_3^4\dot{\pi}^3$ (which comes from the unitary gauge operator $(g^{00}+1)^3$) results in

$$f^{\rm equil}_{{\rm NL},\dot{\pi}^3} \sim 1 - \frac{4}{3}\frac{M_3^4}{M_{\rm Pl}^2|\dot{H}|c_2^{-2}} \tag{4.41}$$

We will later extend this to the four-point function (and in theory, even higher order correlators can be explored).

In passing, we should note that estimates for the unitarity cutoff can be calculated. To summarize the results, for $\dot{\pi}(\partial_i\pi)^2$, the cutoff is given by

$$\Lambda^4_{\dot{\pi}(\partial_i\pi)^2} \sim 16\pi^2 M_{\rm Pl}^2|\dot{H}|\frac{c_s^5}{(1-c_s^2)^2} \tag{4.42}$$

and the cutoff due to $\dot{\pi}^3$ is of the form

$$\Lambda^4_{\dot{\pi}^3} \sim \Lambda^4_{\dot{\pi}(\partial_i\pi)^2}\cdot\frac{1}{(c_s^2+\frac{2}{3}\tilde{c}_3)^2} \tag{4.43}$$

See [96] for further details.

4.5.3 A Note on the Null Energy Condition

Intuitively, H cannot increase because gravity is attractive. We can conclude $\dot{H} \leq 0$ as a consequence of the Null Energy Condition (NEC) [135], which requires that for all null vectors n^μ, the matter stress-energy tensor must satisfy $T_{\mu\nu}n^\mu n^\nu \geq 0$, which implies $\rho + p \leq 0$ and therefore $\dot{H} \leq 0$.

The EFT approach above makes the role of symmetries highly apparent, including the effects of $\dot{H} > 0, \dot{H} = 0$, or $\dot{H} < 0$, elucidating the relationship between inflation, various theories of modified gravity, and theories which do violate the NEC (e.g. bouncing models [87], New Ekpyrotic Cosmology [89], etc.)

For instance, in cases of consistent inflationary models such that the squared sound speed of the fluctuations c_s^2 is negative would allow for a consistent violation of NEC, thereby allowing the possibility of detecting gravitational waves with a blue tilt [21].

4.6 Four-point Function of Non-Gaussianity

So far, we have introduced the three-point function as the most sensitive observable of non-Gaussianities of primordial density perturbations. We previously detailed how single-field inflation can indeed produce a detectable level of non-Gaussianity in the three-point functions (in violations of simple slow-roll).

However, using the results of EFT inflation, we can explore inflationary models where the leading source of non-Gaussianity is the four-point function, or "trispectrum."

Recall we normalized the three-point function of bispectrum to be roughly of the form

$$\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \rangle = \frac{6}{5} \big(P_{\mathbf{k}_1} P_{\mathbf{k}_2} + \text{cyclic terms} \big) \tag{4.44}$$

We may similarly extend this formalism to define a four-point function or trispectrum of the form

$$\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \mathcal{R}_{\mathbf{k}_3} \mathcal{R}_{\mathbf{k}_4} \rangle = \tau_{\mathrm{NL}} \big(P_{\mathbf{k}_1} P_{\mathbf{k}_3} P_{|\mathbf{k}_1 + \mathbf{k}_2|} + 11 \text{ permutations} \big) \tag{4.45}$$

where the amplitude of the trispectrum τ_{NL} is defined as $\tau_{\mathrm{NL}} = \big(\frac{6}{5} f_{\mathrm{NL}}\big)^2$.

The relationship between the three-point and four-point correlation functions is given by the "Suyama-Yamaguchi inequality", given by

$$\tau_{\mathrm{NL}} \geq \left(\frac{6}{5} f_{\mathrm{NL}} \right)^2 \tag{4.46}$$

which has been proven to hold at all levels of expansion and all loop diagrams in inflation [136-137].

Using EFT inflation, we may show it is even possible to generate a detectable four-point function in the absence of a detectable three-point function. This is another incentive for using the effective field theory methods we've been discussing in this chapter. Using EFT Inflation, we may impose an approximate continuous shift symmetry and an approximate parity symmetry on the inflaton fluctuations, which subsequently forbid all cubic terms. The results in a unique quartic operator $\dot{\pi}^4$ where π denotes inflationary fluctuations, and therefore a unique shape of the non-Gaussian four-point function.

Recall there are only two ways to achieve a large three-point function in EFT single-field inflation (such that the Goldstone boson is protected by a continuous shift symmetry): Either the sound speed of the fluctuations is very small (away from the de

Sitter limit) or in the case that the unperturbed solution is so close to de Sitter space $\dot{H} \to 0$ such that the dispersion relation of the Goldstone boson is given by $\omega^2 \sim k^4/M^2$ (where M is some mass scale related to M_2 and $\bar{M}_{2,3}$). The former is found to be

$$f_{\rm NL}\zeta \sim \frac{\mathcal{L}_3}{\mathcal{L}_2}\Big|_{E\sim H} \sim \frac{1}{c_s^2} \quad \Longrightarrow \quad f_{\rm NL} \sim \frac{1}{c_s^2} \tag{4.47}$$

To review, this result implies that in the limit where the scalar perturbations propagate with a small sound speed c_s, this implies large interactions via non-linearly-realized time-symmetries in the Lagrangian, and ergo large primordial non-Gaussianities. This is a quite powerful result: non-Gaussianity increases as c_s decreases, i.e.

$$f_{\rm NL}^{\rm local} \propto 1/c_s^2 \tag{4.48}$$

Now let's consider the four-point function:

The unitary gauge operators $(\delta g^{00})^2$ and $(\delta g^{00})^3$ contain many quartic operators, e.g. the operator $M_2^4(\partial_i\pi)^4$. This operator induces a 4-pt function given by

$$\frac{\langle\zeta^4\rangle}{\langle\zeta^2\rangle^2} \sim \frac{\mathcal{L}_4}{\mathcal{L}_2}\Big|_{E\sim H} \sim \frac{1}{c_s^4}\zeta^2 \tag{4.49}$$

We normally parametrize the four-point function by $\tau_{\rm NL} = \langle\zeta^4\rangle\langle\zeta^2\rangle^3$ such that

$$\tau_{\rm NL}\zeta^2 \sim \frac{\mathcal{L}_4}{\mathcal{L}_2}\Big|_{E\sim H} \sim \frac{1}{c_s^4}\zeta^2 \Longrightarrow \tau_{\rm NL} \sim \frac{1}{c_s^4} \tag{4.50}$$

Note that in order for a four-point function to be detectable,the value of $\tau_{\rm NL}$ has to be a factor of 10^5 larger than the value of $f_{\rm NL}$ allowed by the data. Current limits from WMAP [15] set $f_{\rm NL} \lesssim 10^2$ which would imply that $c_s^2 \gtrsim 10^{-2}$. Therefore current limits on $\tau_{\rm NL}$ are expected to be of the order 10^7. Even a detection of $f_{\rm NL}$ at its current upper bound of 10^2 would result in $\tau_{\rm NL} \sim 10^4 - 10^5$ which will not be detectable by Planck, though future 21-cm experiments with the potential to map a large fraction of our Hubble volume may reach $f_{\rm NL} \sim 10^{-2}$, $\tau_{\rm NL} \sim 10^3$ [138].

Note that the reason the induced four-point function was considered too be so small previously was due to the fact that coefficients of the quadratic operators induced by $(\delta g^{00})^2$ and $(\delta g^{00})^3$ were tied to the ones of cubic operators, a result of non-linear realization of time-diffeomorphisms.

Consider an approximate $\mathbb{Z}_2$ parity symmetry $\pi \to \pi$ on operators $(\delta g^{00})^2$ and $(\delta g^{00})^3$: it is easy to see that these operators do not respect such a symmetry because they contain even and odd powers of π respectively.

Let's identify this symmetry which induces such a four-point function. We begin by setting operators $(\delta g^{00})^2$ and $(\delta g^{00})^3$ to zero, as they can be shown to be suppressed at higher energies than $(\delta g^{00})^4$ for $\tau_N \gtrsim 1$. By reintroducing the field Goldstone field π as we've previously discussed, we find the terms

$$M_4^4(\delta g^{00})^4 \;\rightarrow\; M_4^4\big(16\dot{\pi}^4 - 32\dot{\pi}^3(\partial_\mu\pi)^2 + 24\dot{\pi}^2(\partial_\mu\pi)^4 - 8\dot{\pi}(\partial_\mu\pi)^6 + (\partial_\mu\pi)^8\big) \tag{4.51}$$

Note the $\dot{\pi}^4$ term will induce a 4-pt function of the size

$$\frac{\langle\zeta^4\rangle}{\langle\zeta^2\rangle^2} \sim \tau_{\rm NL}\zeta^2 \sim \frac{\mathcal{L}_4}{\mathcal{L}_2}\Big|_{E\sim H} \sim \frac{M_4^4}{\dot{H}M_{Pl}^2}\zeta^2 \implies \tau_{\rm NL} \sim \frac{M_4^4}{\dot{H}M_{Pl}^2} \tag{4.52}$$

Here we expect the terms $(\delta g^{00})^2$ and $(\delta g^{00})^3$ to be generated. However it can be shown (see [139] for details) that these operators are suppressed via an approximate parity symmetry $\pi \rightarrow -\pi$ and an approximate shift symmetry on the inflaton fluctuations. In such a case, the value of $f_{\rm NL}$ would just be of order $\mathcal{O}(1)$ while $\tau_{\rm NL}$ could be of order $\mathcal{O}(10^5)$! This large of a four-point function is in fact only produced by the single operator $\dot{\pi}^4$ and therefore is of a unique shape. Such a configuration is peaked in momentum space such that all momenta have similar wavelengths.

This result implies $\tau_{\rm NL}$ may be of the order $\tau_{\rm NL} > \mathcal{O}(10^6)$, which is detectable with the WMAP data. However, to the best our of knowledge, the current WMAP 5-yr data has not been analyzed for such a signal. Furthermore, the theory and analysis of a four-point function in the CMB data has largely been overlooked. In general, it is interesting to consider the theoretical work which could be done to further classify inflationary models via non-Gaussian four-point functions.

To summarize, EFT Inflation has allowed us to detect single-field models which generate a detectable four-point function in the absence of a three-point function, an observational window into inflaton previously unexplored. Such a result could have powerful observational ramifications.

4.7 Non-Gaussianities in Single Field Inflation

(Unfortunately, the explicit details of this analysis is beyond the scope of this paper. We will however detail the most current observational constraints below.)

Recall that we have just described a generalized description of all single-field models with a single action in terms of the fluctuations of the inflaton. Furthermore, recall that in the simplest slow-roll scenario, the three-point correlation function is projected to be nearly zero.

That is, if adiabatic curvature perturbations $\zeta_{\mathbf{k}}$ were exactly Gaussian, then all statistical properties would be encoded in the two-point function. Such slow-roll conditions imply $f_{\rm NL}^{\rm local} \sim (\frac{5}{12})(1-n_s)$ is far too small to be detectable (Maldacena 2003). So, non-Gaussian signals in the squeezed limit produced by single-field inflation must be proportional to the tilt of the power spectrum [99,100]. Therefore, barring a detection of strong deviation from scale invariance in the power spectrum, detection of this variety of non-Gaussianity would rule out all single-field inflationary models.

However, much theoretical work has been done recently to extract non-Gaussian predictions from non-slow-roll single field inflation, while the promise of observational precision make such measurements tangible. Current experiments have constrained the level of non-Gaussianity such that any deviation from Gaussian statistics is at most at the 1% level.

Recall that we introduced in Chapter 3 two types of non-Gaussian signals: $f_{\rm NL}^{\rm local}$ where the size of the signal is peaked on squeezed triangle configurations, and $f_{\rm NL}^{\rm equil.}$, where the size of the signal is peaked on equilateral triangle configurations.

The most recent constraint on the $f_{\rm NL}^{\rm local}$ signal is found via WMAP 5-year data to be [115]

$$-4 \leq f_{\rm NL}^{\rm local} \leq 80 \quad \text{at} \quad 95\% \quad \text{CL} \tag{4.53}$$

(Combined with constraints from LSS which we have not discussed, this gives $-1 \leq f_{\rm NL}^{\rm loc.} \leq 63$ at 95% CL).

The most recent constraint on the $f_{\rm NL}^{\rm equil.}$ signal [116] has been found to be

$$-125 \leq f_{\rm NL}^{\rm equil.} \leq 435 \quad \text{at} \quad 95\% \quad \text{CL} \tag{4.54}$$

Needless to point out, we have so far found no evidence for non-Gaussianity.

However, with the insight of EFT inflation, we can parametrize all possible signatures of inflation, particular previously overlooked ones. That is, it has been argued that the parameter space of the non-Gaussianities produced by the most general single-field models where the inflation fluctuation have an approximate shift symmetry is larger than the one characterized by $f_{\rm NL}^{\rm equil.}$. It is rather a linear combination of two independent shapes: the equilateral shape and a shape we will refer to as the "orthogonal" shape. The latter is peaked at both equilateral-triangle configurations and on folded-triangle configurations. As the sign in these two limits are opposite, the new shape is orthogonal to the equilateral one. Due to the approximate shift symmetry of inflaton fluctuations, the most general primordial bispectrum is in fact a linear combination of two independent shapes

$$F(k_1, k_2, k_3) = f_{NL}^{\text{equil.}} F_{\text{equil.}}(k_1, k_2, k_3) + f_{NL}^{\text{orthog.}} F_{\text{orthog.}}(k_1, k_2, k_3) \tag{4.55}$$

where the "orthogonal" bispectrum is defined in Fourier space as

$$F_{\text{orthog.}}(k_1, k_2, k_3) \sim F_{\text{equil.}}(k_1, k_2, k_3) + \frac{1}{k_1^2 k_2^2 k_3^2} \tag{4.56}$$

After expanding the parameter space of non-Gaussian signatures and performing the optimal analysis [116], the most recent constraint on the $f_{\text{NL}}^{\text{local}}$ signal is found to be

$$-4 \leq f_{\text{NL}}^{\text{orthog.}} \leq 80 \quad \text{at} \quad 95\% \quad \text{CL} \tag{4.57}$$

thus finding no evidence of a non-zero $f_{\text{NL}}^{\text{orthog.}}$.

However, the truly remarkable result is the ability to use the EFT formalism previously developed to constrain and even measure the coefficients corresponding to inflaton fluctuations of the interaction Lagrangian. (Such a method has been used previously in particle physics to constrain higher-dimensional operators from precision electroweak tests [128,129]).

Indeed, such an analogy emphasizes the importance of the EFT Inflation method, as it is highly unlikely we could ever produce an inflaton; measuring the energy scale at which inflation happens via the Lagrangian would be a tremendous achievement. Such an effective action has the potential to to provide a complete description of future experimental data!

For example, without getting into too many details concerning the analysis itself (as this would be beyond our paper), it is possible to constrain other parameters in the EFT Lagrangian. For the shape $F_{\dot{\pi}(\partial_i \pi)^2}$ generated by the operator $\dot{\pi}(\partial_i \pi)^2$, $F_{\dot{\pi}^3}$ generated by operator $\dot{\pi}^3$, and $F_{(\partial_j^2 \pi)(\partial_i \pi)^2}$ generated by operator $(\partial_j^2 \pi)(\partial_i \pi)^2$, we can calculate the non-Gaussianity induced by these operators with the bispectrum

$$\langle \Phi_{\vec{k}_1} \Phi_{\vec{k}_2} \Phi_{\vec{k}_3} \rangle = (2\pi)^3 \delta^{(3)} \Big(\sum_i \vec{k}_i \Big) \Big(F_{\dot{\pi}(\partial_i \pi)^2} + F_{\dot{\pi}^3} + F_{(\partial_j^2 \pi)(\partial_i \pi)^2} \Big) \tag{4.58}$$

where $\Psi = \frac{3}{5}\zeta$ such that ζ is the curvature perturbation.

The values of f_{NL} are then given by

$$
\begin{aligned}
f_{\rm NL}^{\dot\pi(\partial_i\pi)^2} &= -\frac{85}{324}\cdot\frac{1}{c_s^2} \simeq -2.662\cdot 10^3\cdot\frac{1}{d_1^{8/5}}\\
f_{\rm NL}^{\dot\pi^3} &= -\frac{10}{243}\Big(\frac{\tilde c_3}{(d_2+d_3)^4}+\frac{3}{2}\Big) \simeq -4.115\cdot 10^{-2}\cdot\frac{\tilde c_3}{(d_2+d_3)^4}\\
f_{\rm NL}^{(\partial_j^2\pi)(\partial_i\pi)^2} &= -\frac{65}{162}\cdot\frac{1}{c_s^2} \simeq -4.072\cdot 10^3\cdot\frac{1}{d_1^{8/5}}
\end{aligned}
\tag{4.59}
$$

Note there are three operators generating non-Gaussianities but only two independent coefficients. These can be shown writing the EFT Lagrangian eq. (4.1) in terms of c_s, such that d_1 disappears, leaving us with

$$
M^4\big(\dot\pi^2 - c_s^2(\partial_i\pi)^2\big) + M^4\dot\pi(\partial_i\pi)^2 + M^4\frac{\tilde c_3}{(d_2+d_3)}\dot\pi^3 + M^4\frac{c_s^2}{H}(\partial_j^2\pi)(\partial_i\pi)^2 \tag{4.60}
$$

A time derivative contributes a factor of H, while a spatial derivative contributes H/c_s, because non-Gaussianities are generated when modes cross the horizon, which implies operators $\dot\pi(\partial_i\pi)^2$ and $(\partial_j^2\pi)(\partial_i\pi)^2$ give rise to an $f_{\rm NL}$ which is parametrically the same as $\mathcal{O}(1/c_s^2)$.

Thus, depending on which operator dominates at horizon crossing, there are two independent shapes on non-Gaussianities. Note that without considering the orthogonal shape, setting these bounds for c_s would have required a set of strong assumptions. Readers are encouraged to read further details found here [116].

Therefore, with EFT inflation, we can map the constraints on the two $f_{\rm NL}$ parameters above into constraints on the coefficient of the interaction Lagrangian of the Goldstone boson. Under the assumption that the primordial density perturbations are generated by a single-field inflationary model where there is an approximate shift symmetry for the Goldstone boson, this mapping is unique and constrains all inflationary models of this kind.

We again stress that no assumption that the background solution is given by a fundamental scalar field: the Lagrangian for the fluctuations in terms of the Goldstone boson is independent of the details through which the background solution is generated; we only assume there is one light degree of freedom playing a relevant role during the inflationary phase.

The speed of sound of the inflaton fluctuations c_s is then constrained to be larger than:

$$c_s \geq 0.011 \quad \text{at} \;\; 95\% \;\; \text{CL} \tag{4.61}$$

or smaller than

$$c_s \lesssim 10^{-2}(d_2 + d_3)^{2/5} \tag{4.62}$$

where the higher-derivative kinetic term proportional to $(d_2 + d_3)$ is important at horizon crossing and the non-Gaussianities depend on other coefficients.

Observe the cases such that inflationary models have a negative squared speed of sound c_s^2 for the fluctuations at horizon crossing. This would lead to an increased level of non-Gaussianities due to the exponential growth of the perturbations before horizon crossing. We may practically rule these out with current analyses from the WMAP data at 95% CL.

4.8 Summary

In this chapter we have introduced the Effective Field Theory of Inflation, a formalism which unifies all single-field models, allowing us to explore the signatures of the dynamics in full generality via operators, including previously overlooked signals (e.g. a detectable four-point function signal in the absence of a three-point function signal). This in fact increases the parameter space by which non-Gaussian signals may be discovered. Moreover, by constraining the terms of the action via cosmological observations, this results in a Lagrangian of the fluctuations written in terms of observational data. Such an approach is systematic, and does not directly rely upon the distinct physics of the inflationary scenario. In contrast to discussing inflationary potentials and kinetic terms, we can characterize inflation solely in terms of observables.

Chapter 5

Future Prospects and Conclusions

5.1 Future Observational Prospects

We discussed at the end of Chapter 2 efforts to constrain inflationary models with more precise measurements could possibly detect deviations from adiabatic and scale-invariant density fluctuations. In the former case, a violation of adiabaticity could point to the existence to inflationary models with more than one scalar field, a signature of mutlifield inflation (see Appendix A for further details).

The most distinct observation in favor of the inflationary paradigm would be robust detection primordial $B-$modes, as a background of gravitational waves is seen by cosmologists as the "smoking gun" for inflation. Several upcoming ground-based and balloon experiments (e.g. Planck, CMBPol, etc.) are expected to be of the capacity to detect such a signal [140,141].

Furthermore, this paper has been devoted to classifying and constraining inflationary models on the basis of deviations from Gaussian density perturbations. However, all CMB and LSS probes in the near future (e.g. Planck) are expected not to be sensitive enough to detect the nearly Gaussian signal predicted by simple slow-roll, i.e. $f_{\rm NL} \sim \mathcal{O}(10^{-2})$. We will now discuss several ideas which may have the capability to probe such scales:

5.1.1 CMB $\mu-$distortions

Let's begin by reminding ourselves about a few details of CMB physics. At early times $z \gg 10^6$, when free electrons were coupled to photons and baryons via Compton scattering, the photon-baryon plasma was a nearly perfect fluid. If there are any perturbations of the full thermodynamic equilibrium between photons and baryons, spectral distortions in the CMB will result. For example, the Sunyaev-Zel'dovich distortion [142] is due to photons from the CMB scattering through the hot gas of galaxy clusters by rapidly moving electrons. At such times $z \gg 10^6$, the frequency of photon scattering via Compton scattering was so efficient that maintaining thermal equilibrium ("thermalization") was nearly perfect. Before this redshift however the CMB may be subject to certain spectral distortions which may be observable today. We are interested in $\mu-$distortions [143,144] which characterize higher-redshifts (i.e. $z \geq 10^5$), where the number density of photons (ν) is given by the Bose-Einstein distribution

$$n(\nu) = \left[e^{x+\mu(\nu)} - 1\right]^{-1} \tag{5.1}$$

where $\mu(\nu)$ is the frequency-dependent chemical potential. See [145] for more details.

In such a case, due to the dissipation of acoustic waves of the adiabatic mode (i.e. "Silk damping") [146], we could possibly use μ-distortion to constrain the primordial power spectrum. Such a mechanism is estimated to be far more sensitive than current techniques, as it could probe primordial perturbations which have now been completely erased by Silk damping.

Furthermore, it has been estimated in [147,148] that such a probe could possibly reach a squeezed-limit signal of $f_{\rm NL}^{\rm local} \sim \mathcal{O}(10^{-3})$, a result which could very well measure inflationary dynamics up to the simple slow-roll limit.

5.1.2 21-cm Tomography

The next frontier of observational cosmology is study the epoch of the dark ages (i.e. from the last scattering surface of the CMB to the formation of luminous structures, between $z \sim 1000$ and $z \sim 6$). Perhaps the most promising probe discussed today is to study the redshifted 21-cm line of hydrogen at $z = 1 - 150$. Indeed, a 21-cm tomography of a three-dimensional mapping of neutral hydrogen over such a wide range of redshifts could prove to be a more powerful probe than the CMB or galaxy surveys, with a possible detection capability which could improve the precision of cosmological parameters to an unprecedented level (see [149] for further details). Thus, any detected deviation from the expected values would be a signature of new physics.

Roughly, the thermal evolution of cosmic gas was coupled to the CMB down to a redshift $z \sim 200$; after this point, the gas temperature dropped adiabatically as

$T_{\rm gas} \propto (1+z)^2$ below the temperature of the CMB, $T_\gamma \propto (1+z)$. At the epoch of reionization (EOR) $z \leq 200$, the first stars emitted photo-ionizing ultraviolet light, thus heating the gas up again. Hence, before and after the formation of the first stars, cosmic neutral hydrogen absorbed the CMB flux through its 21-cm transition. The inhomogeneous density distribution of neutral hydrogen therefore creates anisotropies in the brightness temperature measured with respect to the near blackbody CMB. Thus the neutral hydrogen 21-cm line is the most direct probe of luminous structure formation at $6 < z < 11$.

As a result of the magnetic moments of the proton and the electron, there exists a hyperfine spin-flip transition of neutral hydrogen from the 1S ground state, producing two distinct energy levels separated by $\Delta E = 5.9 \times 10^{-6}$ eV corresponding to a wavelength $\lambda = 21$ and frequency 1420 MHz.

It follows then that the results from 21-cm tomography have the potential to powerfully constrain inflationary models. Much work has been done recently to forecast the strength and accuracy by which 21-cm physics could parametrize inflationary models. This includes:

- Using 21-cm data in conjunction with future Planck data to constrain the primordial scalar spectral index n_s and tilt α_s, which would tightly confine slow-roll parameters, thus determining the shape of the inflationary potential [138,150].

- Recall the confusion associated with detecting $B-$mode polarization due to gravitational lensing (see Appendix A). 21-cm tomography could possibly separate such lensing-induced CMB polarization anisotropies from inflationary gravitation waves using anisotropies in the cosmic 21-cm radiation. B-modes may also be produced by Faraday rotation of the 21-cm emission via the galactic foreground emission [151].

  Not only would this provide a clean signature for inflation, but together with additional data from CMB polarization, one could roughly probe inflationary energy scales (barring recent considerations which source gravitational waves via other mechanisms [63]).

- Furthermore, such 21-cm experiments are forecasted to be powerful enough to detect non-Gaussian deviations from the angular power spectrum at a far greater sensitivity than currently known [138].

If primordial fluctuations were indeed non-Gaussian, then 21-cm anisotropies would consequently contain such signature. In fact, this seems to be our most promising approach of probing and refining any non-Gaussianity associated with single-field slow-roll beyond Planck, as we may be able to limit the primordial non-Gaussianity parameter to $f_{\mathrm{NL}} \leq 0.01$

Note that the development of feasible 21-cm detectors is still a burgeoning field. There remains a number of experimental challenges today for future 21-cm measurements, mainly the foreground interference from man-made terrestrial radio broadcasting, extragalactic radio sources, etc. Such measurements remain an unparalleled prospect to precisely constrain inflationary models.

(See reviews [152,153] for a comprehensive review of 21-cm tomography physics and experimental development, which is outside the scope of this paper.)

5.2 Future Theoretical Work

As we discussed in Chapter 2, to realize the full potential of using non-Gaussianity to constrain and possibly falsify inflationary models, theorists must systematically account for all effects inducing observable levels of non-Gaussianity per inflationary model. Moreover, these predictions must take into account the non-linear effects of the perturbations. In this section, we summarize by discussing further ways to fortify our knowledge of this problem, and further ideas to possibly constrain inflation.

5.2.1 Effective Field Theory Methods

It has been recently discussed as to how to extend the EFT formalism of single-field inflation to include additional degrees of freedom, i.e. to construct an EFT theory of multifield inflation. However, such a construction is highly non-trivial: firstly, there are many candidates of these additional light degrees of freedom, and in some models they are not required to be in the vacuum at the moment of horizon crossing. Even in the case when such fields are in the vacuum, it is difficult to have naturally light scalar fields (i.e. lighter than the Hubble length, so as to acquire long-wavelength fluctuations). Such cases rely upon Supersymmetry, or are (pseudo-)Goldstone bosons of a global (non-)Abelian symmetry which is spontaneously broken. Secondly, in contrast to EFT single-field inflation, such a Lagrangian would be much less highly constrained by the symmetries. We recommend readers to consult [154,155] for further discussion and preliminary attempts to tackle this problem.

Correspondingly, there have been efforts to extend the consistency relation of the three-point function in single-field inflation [99,100] for single-field models with additional degrees of freedom such that the additional degrees of freedom do not produce significant contributions to the density perturbations ζ [156]. Further work should be done to explore this relationship, which could improve in an improvement of the above EFT Multifield formalism.

Furthermore, the prospect of continuing to apply effective field theory methods to cosmological issues presents itself as an exceptionally promising endeavor. Indeed, such techniques have recently been introduced to study cosmological perturbations [123] and large scale structures [124].

One fascinating proposal is to construct an EFT theory of single field quintessence. As in the EFT Inflation case, this would allow us to generalize all of single-field quintessence models in order to study the phenomenological consequences. The potential of this formalism applied to this question of cosmology and others is extremely exciting.

5.2.2 Chiral Gravity

Recall that cross-correlations between temperature and $E-$ and $B-$modes come in six varieties: $\langle TT\rangle, \langle EE\rangle, \langle BB\rangle, \langle TE\rangle, \langle TB\rangle$, and $\langle EB\rangle$.

The $E-$mode and $B-$mode power spectra may be written as

$$\langle a_{lm}^{E} a_{l'm'}^{*E}\rangle = \delta_{ll'}\delta_{mm'}C_l^{EE} \tag{5.2}$$

$$\langle a_{lm}^{B} a_{l'm'}^{*B}\rangle = \delta_{ll'}\delta_{mm'}C_l^{BB} \tag{5.3}$$

where we observe that the $E-$mode and $B-$mode patterns behave differently under parity transformations $\hat{\mathbf{n}} \to -\hat{\mathbf{n}}$ such that $B-$patterns change signs

$$a_{lm}^{E} \to (-1)^l a_{lm}^{E} \tag{5.4}$$

$$a_{lm}^{B} \to (-1)^{l+1} a_{lm}^{B} \tag{5.5}$$

For these symmetry reasons, $\langle TB\rangle$ and $\langle EB\rangle$ correlations vanish, i.e.

$$\langle a_{lm}^{E} a_{l'm'}^{*B}\rangle = 0 \tag{5.6}$$

However, if gravity in fact violated parity, this would not be true and we would expect a non-vanishing $\langle TB\rangle$ correlator. In fact, detecting a $\langle TB\rangle$ signal would actually be a signature for both a gravitational wave background and chiral gravity (in contrast to a detection of $\langle BB\rangle$, which solely affirms the former).

(There are several motivations for considering the possibility that gravity is party-violating. As the weak sector of the Standard Model violates parity, it is natural to question whether P-violation may be relevant for inflation [or late-time acceleration]. Furthermore, the Cartan-Kibble formulation [157] of General Relativity as well as the Ashtekar formalism [158] show gravity has the capacity for chirality due to terms in the action manifesting odd parity. See [165-167] for details.)

In consideration of non-Gaussian corrections to primordial fluctuations from inflation, we previously studied and classified shapes of correlation functions from scalar fluctuations (as in Appendix C, eq. (132)). Analogously, by computing the general two-point function, we find

$$\langle h^{s1}_{\mathbf{k}} h^{s2}_{\mathbf{k}'}\rangle = (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')\frac{1}{2k^3}\left(\frac{H_*}{M_{\rm Pl}}\right)^2 4\delta_{s_1,s_2} \tag{5.7}$$

However, if gravity was parity violating, resulting in left L and right R gravitons, such parity breaking terms result in a different amplitude for left and right gravitational waves, leading to a neat circular polarization. Recall that in our discussion of inflation, if the two-point correlation function above does not contain all information, higher order correlation functions come in to play, describing non-Gaussianity. As before, we primarily measure such non-Gaussianity with a three-point function/bispectrum, calculated to be [159-161]

$$\begin{aligned}\langle h^{s1}_{\mathbf{k}_1} h^{s2}_{\mathbf{k}_2} h^{s3}_{\mathbf{k}_3}\rangle =&(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\left(\frac{H_*}{M_{\rm Pl}}\right)^4\frac{4}{(2k_1k_2k_3)^3}\times\\ &[(k_i^2k_j^2\epsilon^1_{ij})\epsilon^2_{kl}\epsilon^3_{kl} - 2\epsilon^1_{ij}(k_l^3\epsilon^2_{li})(k_m^2\epsilon^3_{mj}) + \text{cyclic}]\ \times\\ &\left(k_1+k_2+k_3 - \frac{k_1k_2+k_1k_3+k_2k_3}{k_1+k_2+k_3} - \frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right)\end{aligned} \tag{5.8}$$

Indeed, this means that inflationary models which give rise to non-Gaussianity could also give rise to parity breaking contributions, i.e. circularly polarized gravitational waves. (For example, single-field slow-roll models with higher derivative terms would cause parity breaking contributions in the two-function, etc.). Therefore, without assuming linear gravitational, the tensor spectrum of fluctuations during inflation may expose signatures of inflationary physics associated with chiral gravity.

Thus, we should be able to classify and possibly constrain inflationary models by signatures of chiral gravitational waves. However, this result has not been rigorously explored for the myriad of inflationary models. In fact, there exists the possibility that the signature of asymmetry between left- versus right-handed gravitational waves could differentiate between inflationary models which may have similar non-Gaussian signatures (and other features). Much more theoretical work must be done to investigate this possible (as to our knowledge, a comprehensive library of how each class of inflationary models possibly violate parity hasn't been accomplished previously).

Furthermore, there is still much observational effort to detect such signals of the gravitational radiation (e.g. Spider [162], EBEX [163], Planck [140], CMBPol [141], etc.). The sensitivity of these respective experiments to detect such signals has recently been evaluated in [164].

5.3 Falsifying Inflation

As we have detailed, so far no deviations from Gaussian fluctuations have been found. However, with the arrival of upcoming experimental and further theoretical insight, the potential to constrain and falsify inflationary models is highly promising.

As we asserted with the use of the EFT single-field formalism, it is possible for single-field models to exhibit a distinct four-point signal in the absence of a three-point signal. As far as we are aware, a rigorous analysis on existing WMAP data has yet to be performed to search for such a signal. Moreover, the prospect of searching for such a signal using data from future, more precise experiments is very encouraging.

In consideration of the theoretical bounds detailed previously, the non-Gaussian squeezed limit signal is proportional to the tilt of the power spectrum. Recall that single-field models cannot produce a signal $f_{\rm NL}^{\rm local}$ greater than the deviation of the power spectrum from scale invariance [99,100]. Therefore, a non-Gaussian detection of $f_{\rm NL}^{\rm local} > 1$ would confidently rule out all single-field inflationary models (barring a significant deviation of scale invariance of the power spectrum). Furthermore, using the Suyama-Yamaguchi $\tau_{\rm NL} \geq (6/5 f_{\rm NL})^2$ inequality discussed in Chapter 3, a significant detection of $f_{\rm NL}$ with a non-detection of $\tau_{\rm NL}$ would practically rule out both single-field *and* multifield models. Such a result would be a serious blow to the inflationary paradigm, and would certainly impel cosmologists to rethink our interpretation of the very early universe.

Appendix A

The first section of this appendix largely deals with the motivation and background for inflation, and the second section gives a background to CMB physics and how it relates to inflation. This appendix is meant to provide sufficient background in cosmology, with few details and even fewer derivations. We recommend referring to any current introductory cosmology textbook for further (and more cohesive) information (e.g. Dodelson's *Modern Cosmology* [168]).

Background Cosmology and Motivating Inflation

Inflation is a period of exponential expansion in the early universe (10^{-34} seconds after the Big Bang singularity) considered to be responsible both for large-scale homogeneity of the universe and for the small fluctuations that were the seeds of structure formation.

It's undeniable that the universe is expanding from BB, with evidence ranging from light from distant galaxies redshifted, observed abundances of the predicted light elements (H, He, and Li) from BBN (Big Bang Nucleosynthesis).

FRLW Cosmology

Assuming homogeneity and isotropy on large scales leads us to the FLRW metric for the space-time of the universe, where $a(t)$ is the scale factor characterizing the relative size of hypersurfaces Σ at different times, curvature parameter k is $+1$ for positively curved Σ, 0 for flat Σ and -1 for negatively curved Σ.

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\Big) \tag{9}$$

or, defining $d\chi^2 = \frac{dr^2}{1-kr^2}$, as

$$ds^2 = -dt^2 + a^2(t)\Big(d\chi^2 + \Phi_k(\chi^2)(d\theta^2 + sin^2\theta d\phi^2)\Big) \tag{10}$$

where

$$r^2 = \Phi_k(\chi^2) \equiv \begin{cases} \sinh^2\chi & k = -1 \\ \chi^2 & k = 0 \\ \sin^2\chi & k = -1 \end{cases} \tag{11}$$

The Hubble parameter H is the expansion rate of FLRW spacetime in units of inverse time $H \sim t^{-1}$ and is positive for an expanding universe. It sets the scale for the age of the universe, and the Hubble distance sets the size of the observable universe

$$H \equiv \frac{\dot{a}}{a} \tag{12}$$

Observe that this is not in reality a constant but changes with time. Today, we've constrained the Hubble rate to be $H_0 \sim 72 \ \text{km s}^{-1}\text{Mpc}^{-1}$

Conformal Time and Null Geodesics

In discussing the causal structure of the universe, observe that massless photons follow null geodesics $ds^2 = 0$. In order to discuss the maximum comoving distances light can propagate, we define conformal time

$$\tau = \int \frac{dt}{a(t)} \tag{13}$$

Using this, we define the particle horizon as the maximum comoving distance light can propagate between an initial time t_i (often taken at the origin $t_i \equiv 0$) and some later time t, giving

$$\chi_p(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)} \tag{14}$$

with distance of the particle horizon defined by $d_p(t) = a(t)\chi_p$. This definition is vital to understanding the various problems of the early Universe, as it imposes limits in the past where spacetime could have been in causal contact.

Similarly, the event horizon defines the boundary at which signals at a time τ will never be received by an observer in the future

$$\chi > \chi_e = \int_{\tau}^{\tau_{\max}} \mathrm{d}\tau = \tau_{\max} - \tau \tag{15}$$

with size of the event horizon $d_e(t) = a(t)\chi_e$.

Einstein Equations

The dynamics of the FLRW universe with scale factor $a(t)$ is governed by the Einstein Equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{16}$$

Recall the definition of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{17}$$

in terms of Ricci tensor $R_{\mu\nu}$ and Ricci scalar R,

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{18}$$

where

$$\Gamma^{\mu}_{\alpha\beta} \equiv \frac{g^{\mu\nu}}{2}\left[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}\right] \tag{19}$$

where commas denote partial derivatives, e.g. $(...)_{,\mu} = \frac{\partial(...)}{\partial x^{\mu}}$.

In the case of a perfect fluid (i.e. described entirely in terms of energy density ρ and isotropic pressure p) with the metric above, we may define the energy-momentum tensor $T^{\mu\nu}$

$$T^{\mu\nu} = (\rho + P) u_{\mu} u_{\nu} + p g_{\mu\nu} \tag{20}$$

where u^{μ} is the four-velocity of the observer (corresponding to a fluid-element).

With this assumption, we may derive the Friedman equations, which govern the expansion of FLRW spacetime in terms of General Relativity

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2}, \qquad \dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) \tag{21}$$

These may be combined to give us the continuity equation

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0 \tag{22}$$

which may be written as $\frac{d\ln\rho}{d\ln a} = -3(1+\omega)$. If we define the equation of state parameter $\omega = \frac{p}{\rho}$, this us gives $\rho \propto a^{-3(1+\omega)}$

Together with the Friedman equations eq. (21), this gives the time evolution of the scale factor

$$a(t) \propto \begin{cases} t^{2/3(1+\omega)} & \omega \neq -1 \\ e^{Ht} & \omega = -1 \end{cases} \tag{23}$$

such that $a(t) \propto t^{2/3}, a(t) \propto t^{1/2}$, and $a(t) \propto \exp(Ht)$ respectively for the a flat ($k = 0$) universe dominated by non-relativistic matter/radiation ($\omega = 0$) or relativistic matter ($\omega = \frac{1}{3}$) and a cosmological constant ($\omega = -1$).

For more than one matter species (baryons, photons, neutrinos, dark matter, dark energy, etc.) contributing significantly to the energy density, and the pressure, ρ and p refer to the sum of all components

$$\rho \equiv \sum_i \rho_i, \qquad p \equiv \sum_i p_i \tag{24}$$

where for each species i we defined the present ratio of the energy density relative to the critical energy density $\rho_{\text{crit}} \equiv 3H_0^2$

$$\Omega_i \equiv \frac{\rho_0^i}{\rho_{\text{crit}}} \tag{25}$$

and the corresponding equations of state $\omega_i \equiv \frac{p_i}{\rho_i}$

Problems with the Big Bang Model

While there are a multitude of theoretical issues and mysteries surrounding the Standard Big Band Model, we will discuss the two major motivations for the inflationary paradigm, the "Horizon Problem" and the "Flatness Problem", which are at their core problems of initial conditions. [1]

[1] We should be clear that these issues are strictly speaking not inconsistencies of the Big Bang. If we were to simply assume that the universe was flat with an initial value of $\Omega = 1$, and that the universe began homogeneously over superhorizon distances (with the exact level of inhomogeneity to account for structure formation), the early universe will evolve into our FLRW universe.

- **The Horizon Problem**

  Expressing the comoving particle horizon τ between time 0 to t in terms of the comoving Hubble radius $(aH)^{-1}$, we find

$$\tau \equiv \int_0^t \frac{dt'}{a(t'0} = \int \frac{da}{Ha^2} = \int_0^a d\ln a\Big(\frac{1}{aH}\Big) \tag{26}$$

  For a universe dominated by a fluid with equation of state ω, this gives us

$$\big(aH\big)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3\omega)} \tag{27}$$

  Note that this implies as the comoving Hubble radius grows, the comoving horizon τ (i.e the fraction of the universe in causal contact) increases with time

$$\tau \propto a^{\frac{1}{2}(1+3\omega)} \tag{28}$$

  where this qualitative behavior depends on whether $(1+3\omega)$ is positive or negative. Observe this would mean that comoving scales entering the horizon today have been far outside the horizon at CMB decoupling (where the comoving wavelength of the fluctuations is time-indepdent, and the comoving Hubble radius is time-dependent). However, the CMB is homogeneous to the degree of one part in 10^4, which implies the unverse was extremely homogeneous at the time of last-scattering. How could regions that *a priori* must be causally independent made these regions so homogeneous?

  The Horizon Problem is even more worrying if we take estimate a causal region at Plank time, giving

$$\frac{d_p(t_{\mathrm{Pl}})}{d_p(t_0)} \approx 10^{-26} \tag{29}$$

  This implies the universe observed today would be made up of around 10^{78} regions causally independent at Plank time t_{Pl}. How could that possibly be?

- **Flatness Problem**

  It is possible to write the Friedman equation in the form

$$1 - \Omega(a) = \frac{-k}{(aH)^2} \tag{30}$$

  where

$$\Omega(a) = \frac{\rho(a)}{\rho_{\text{crit}}(a)}, \qquad \rho_{\text{crit}}(a) \equiv 3H(a)^2 \tag{31}$$

  In standard FLRW cosmology, the quantity $|\Omega - 1|$ must thus diverge with time as the comoving Hubble radius $(aH)^{-1}$ grows with time. This critical value $\Omega = 1$ is an unstable fixed point; the near-flatness observed today $\Omega(a_0) \sim 1$ requires an extreme fine-tuning of Ω close to 1 int he early universe.

  One finds the deviation of flatness from at Big Bang Nucleosynthesis (BBN) to the Planck scale has to satisfy

$$|\Omega(a_{\text{BBN}}) - 1| \leq \mathcal{O}(10^{-16}) \tag{32}$$

$$|\Omega(a_{\text{Pl}}) - 1| \leq \mathcal{O}(10^{-61}) \tag{33}$$

Inflationary Paradigm

The basic idea behind inflation [1-6] is to shrink the comoving Hubble radius sufficiently in the early universe such that the comoving horizon is much greater than the comoing Hubble radius $(aH)^{-1}$, i.e. $\tau \ggg (aH)^{-1}$. Remember, if points are separated by distances greater than τ, then they were never causally dependent. If points are separated by distances greater than $(aH)^{-1}$, then they are not *currently* in causal contact. With this intuition in place, we turn to the conditions of inflation:

- **Decreasing comoving horizon**

  We define the shrinking Hubble sphere as

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \tag{34}$$

  This is referred to as the fundamental definition of inflation as it both most directly relates to the flatness and horizon problems and is key for the mechanism to generate fluctuations.

- **Accelerated expansion**
  Using the relation
$$\frac{d}{dt}(aH)^{-1} = -\frac{\ddot{a}}{(aH)^2} \tag{35}$$
  it follows that a shrinking comoving Hubble radius implies accelerated expansion
$$\frac{d^2 a}{dt^2} > 0 \tag{36}$$
  This is often referred to as the definition of inflation, a period of accelerated expansion. The second time derivative of the scale factor is related to the first time derivative of the Hubble parameter H
$$\frac{\ddot{a}}{a} = H^2(a - \epsilon), \qquad \text{where} \;\; \epsilon \equiv -\frac{\dot{H}}{H^2} \tag{37}$$
  Acceleration therefore corresponds to
$$\epsilon = -\frac{\dot{H}}{H^2} = -\frac{d \ln H}{dN} < 1 \tag{38}$$
  where we have defined $\mathrm{d}N = H\mathrm{d}t = \mathrm{d}\ln 1$, which measures the number of e-folds N of inflationary expansion. This implies the fraction change of the Hubble parameter per $e-$fold is in fact small.

- **Negative pressure**
  The condition $\ddot{a} > 0$, i.e.
$$\frac{\ddot{a}}{a} = -\frac{\rho}{6}(1 + 3\omega) > 0 \tag{39}$$
  requires that pressure be negative, i.e. $p < -\frac{1}{3}\rho$ or $\omega < -\frac{1}{3}$.

In summary, the shrinking comoving Hubble radius $(aH)^{-1}$ relates to the acceleration and the pressure of the universe as follows
$$\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) < 0 \implies \frac{d^2 a}{dt^2} > 0 \implies \rho + 3p < 0 \tag{40}$$
Let's observe how inflation is able to solve the two problems previously mentioned.

For the Horizon Problem, scales of cosmological interest are larger than the Hubble radius H until approximately $a \sim 10^{-5}$. So before inflation, all scales of interest were smaller than the Hubble radius and therefore affected by microphysical processes. At horizon re-entry much later in time, these scales came back within the Hubble radius. The causal physics before inflation therefore established the homogeneity we observed. To summarize, inflation stretches microscopic scales to superhorizon scales, thus correlating spatial regions over distances apparently not in causal contact.

Without inflation, the physical horizon grows faster than the physical wavelengths of perturbations. With inflation, the physical wavelengths grow faster than the horizon.

For the Flatness Problem, recall eq. (30): If the comoving Hubble radius decreases, this drives the universe towards flatness (i.e. $\Omega = 1$)

CMB Physics

This section will offer a bare minimum background to the relevant aspects of the Cosmic Microwave Background. In particular, we will introduce what is considered to be a definite signature for inflation (as opposed to alternatives of the inflationary paradigm): B-mode polarization induced by an inflationary gravitational wave background. For further information and reviews of this rather expansive and detailed subject, see the reviews by [169-171].

First discovered in 1964 [172], the radiation of the cosmic microwave background (CMB) is the thermal afterglow of the recombination era $370,000$ years after the Big Bang, where photons decoupled from matter. Temperature anisotropies (of order $\Delta T/T \sim 10^{-5}$ across $10^\circ - 90^\circ$ on the sky) created from Thomson scattering are due to the spatial variations in the CMB temperature at recombination. On $\sim 1^\circ$ scales, this is associated with acoustic oscillations of the photon-baryon plasma. Furthermore, secondary effects such as gravitational lensing and the Sunyaev-Zel'dovich effect also leave imprints on the CMB. Cosmologists continue to refine observations of the CMB to discover more and more precise effects.

We interpret these temperature fluctuations to be primordial density perturbations from inflation.

Temperature Anisotropies

In order to characterize the spatial and angular distribution, we describe the temperature at position $\hat{\mathbf{x}}$ of the observer in the direction $\hat{\mathbf{n}}$ of the observer's sky. This nearly uniform blackbody spectrum of the radiation may be written with the distribution function

$$f(\nu) = \left[\exp(2\pi\nu/T(\hat{\mathbf{n}}) - 1 \right]^{-1} \tag{41}$$

with observation frequency $\nu = E/2\pi$ and mean temperature of $\bar{T}_0 = 2.725\text{K}$.

Statistically isotropic, Gaussian random temperature fluctuations are expressed in terms of spherical harmonics on a 2-sphere $Y_{lm}(\hat{\mathbf{n}})$ such that

$$\Theta(\hat{\mathbf{n}}) \equiv \frac{\Delta T}{T_0} = \sum_{lm} a_{lm} Y_{lm}(\hat{\mathbf{n}}) \tag{42}$$

such that

$$a_{lm} = \int d\Omega Y^*_{lm}(\hat{n})\Theta(\hat{n}) \tag{43}$$

Note that a_{lm} is statistically independent and there is no $m-$dependence. For Gaussian random fluctuations, the ensemble average of the temperature field is determined by the rotationally-invariant angular power spectrum

$$C_l^{TT} = \frac{1}{2l+1} \sum_m \langle a^*_{lm} a_{lm} \rangle, \quad \text{such that} \ \ \langle a^*_{lm} a_{l'm'} \rangle = C_l^{TT} \delta_{ll'} \delta_{mm'} \tag{44}$$

If the temperature field is of a pure Gaussian distribution, then all statistical properties may be extracted from the power spectrum (i.e. the 2-pt correlation function). Note that as a result of $l+1$ modes existing for each value l, there is a fundamental uncertainty in the information we extract from C_l, called "cosmic variance":

$$\left(\frac{\Delta C_l}{C_l} \right) = \sqrt{\frac{2}{2l+1}} \tag{45}$$

With the use of the power spectrum, we may study the acoustic oscillations to determine features of our universe, such as the geometry of the Universe, various cosmological parameters, and the spectrum of primordial perturbations. Thus, the CMB has conclusively given us confidence in a flat universe with a nearly-scale invariant spectrum of primordial perturbations. Inflation provides the paradigm to explain the flatness of the Universe, why the seemingly disconnected regions are in causal contact, and where the primordial seeds for large-scale structure (LSS) come from.

Note that CMB temperature fluctuations are dominated by scalar modes R at tensor-to-scalar ratios $r < 0.3$, the current constraints. With the use of a transfer function Δ_{Tl}, we may relate the linear evolution of scalar modes R to CMB temperature fluctuations ΔT such that

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \Delta_{Tl}(k) R_{\mathbf{k}} Y_{lm}(\hat{\mathbf{k}}) \tag{46}$$

Using the identity

$$\sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{k}}') = \frac{2l+1}{4\pi} P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') \tag{47}$$

we then relate the primordial spectrum to anisotropies via

$$C_l^{TT} = \frac{2}{\pi} \int k^2 dk \; \underbrace{P(k)}_{\text{Inflation}} \; \underbrace{\Delta_{Xl}(k)\Delta_{Yl}(k)}_{\text{Anisotropies}} \tag{48}$$

Polarization Anisotropies

Not only does the CMB contain temperature anisotropies, but it is also characterized by polarized anisotropies via Thomson scattering of photons by electrons during recombination and reionization. For example, consider a photon traveling in the $\hat{x}$ direction with its transverse electric field making an electron oscillate in the $\hat{y}$ and $\hat{z}$ plane. Radiation is scattered in these directions, resulting in linear polarization. If photons are incident from all directions, this results in polarization averaging to zero. However, let's consider two perpendicular components of scattered light of different temperatures. In this case, a net linear polarization can occur.

We describe this linear polarization in terms of Stokes parameters Q and U and (analogous to temperature multipole moments a_{lm}) orthogonal components E_{lm} and B_{lm}.

$$(Q \pm iU)(\hat{\mathbf{n}}) = -\sum_{lm} (E_{lm} \pm iB_{lm})_{\pm 2} Y_{lm}(\hat{\mathbf{n}}) \tag{49}$$

such that Q and U transform as a spin-2 field under rotation by angle ψ

$$(Q \pm iU)(\hat{\mathbf{n}}) \to e^{\mp 2i\Psi} (Q \pm iU)(\hat{\mathbf{n}}) \tag{50}$$

requiring harmonic analysis in terms of an expansion in terms of spin-2 spherical harmonics.

The $E-$ and $B-$modes completely specify the linear polarization field as analogues of curl-free and divergence-free components of a vector. $E-$ polarization gives polarization vectors radial around cold spots and tangential around hot spots. $B-$ polarization is divergence-free but not curl-free, giving polarization vectors with vorticity around any point on the sky. Their respective power spectra are defined as:

$$\langle a^E_{lm} a^{*E}_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_l^{EE} \tag{51}$$

$$\langle a^B_{lm} a^{*B}_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_l^{BB} \tag{52}$$

allowing using to measure the polarization map

$$\vec{P}(\hat{\mathbf{n}}) = \vec{\nabla} A + \vec{\nabla} \times \vec{B} \tag{53}$$

where $E-$ and $B-$modes transform differently under parity $\hat{\mathbf{n}} \to -\hat{\mathbf{n}}$ such that $B-$patterns change signs

$$a^E_{lm} \to (-1)^l a^E_{lm} \tag{54}$$

$$a^B_{lm} \to (-1)^{l+1} a^B_{lm} \tag{55}$$

Defining the rotationally invariant angular power spectrum

$$C_l^{XY} \equiv \frac{1}{2l+1} \sum_m \langle a^X_{lm} a^Y_{lm} \rangle, \quad X, Y = T, E, B \tag{56}$$

observe that there are six CMB cross-correlations possible: C_l^{TT}, C_l^{EE}, C_l^{BB}, C_l^{TE}, C_l^{TB}, C_l^{EB}.

Note that correlations $\langle TB \rangle$ and $\langle EB \rangle$ vanish due to symmetry under normal assumptions (i.e. unless there are parity-violating processes in the early-universe; a detection of such correlations would be a signature for chiral gravity. See our conclusions in Chapter 5).

The cosmological significance of the CMB polarization is due to the realization that [173,174]

- scalar/density perturbations create only $E-$modes, but no $B-$modes, i.e. produce a curl-free spatial distribution of the polarization field.

- vector/vorticity perturbations create mainly $B-$modes.
- tensor/gravitational wave perturbations create both $E-$modes and $B-$modes i.e a spatial distribution with non-zero curl.

This is a highly non-trivial result: Inflation predicts that quantum excitations of tensor modes produces a nearly scale-invariant stochastic background of gravitational waves [175-178]

$$h_{ij} \sim \frac{H}{M_{Pl}} \tag{57}$$

The same mechanism that stretches the vacuum fluctuations during inflation, seeding structure formation, generates a stochastic background of gravitational waves. With the above knowledge that scalars do not produce $B-$modes while tensors do has prompted observational cosmology to search for $B-$modes as the distinctive signature of inflation. Alternatives to inflation (e.g. ekpyrosis, VSL models, etc.) largely do not predict a significant amplitude of primordial gravitational waves; thus, the detection of $B-$modes has been largely considered conclusive evidence in favor of inflation. Measuring C_l^{BB} where

$$C_l^{BB} = (4\pi)^2 \int k^2 \mathrm{d}k \underbrace{P_h(k)}_{\text{Inflation}} \Delta^2_{Bl}(k) \tag{58}$$

would give us access to information about primordial tensor fluctuations.

For gravitational waves, we write the perturbed FLRW metric as

$$ds^2 = a(\tau)^2\{-d\tau^2 + [\gamma_{ij} + 2h_{ij}(\mathbf{x},\tau)]dx^i dx^j\} \tag{59}$$

where γ_{ij} is the unperturbed flat space metric, and metric perturbation h_{ij} is traceless ($h^i_i = 0$) and transverse ($\partial_i h^{ij} = 0$), leaving two independent degrees of freedom corresponding to the two gravitational-wave polarization.

Thus, the power spectrum of tensor perturbations due to quantum fluctuations from inflation is given as

$$\langle h^+(k)h^+(k')\rangle = \langle h^\times(k)h^\times(k')\rangle = \frac{\Delta_t^2(k)}{2}\delta(\mathbf{k}-\mathbf{k}') \tag{60}$$

Writing the primordial power spectrum in terms of the Hubble parameter H evaluated when CMB scales enter the horizon, we find

$$\Delta_t^2(k) = \frac{32\pi G H^2}{(2\pi)^3 k^3}\Big|_{aH=k} \tag{61}$$

Note that in the case of slow-roll single-field inflation, H changes little when CMB scales enter the horizon. Thus, in terms of the amplitude A_T and tensor spectral index n_T, we estimate

$$\Delta_t^2(k) = A_T k^{n_T - 3} \tag{62}$$

where slow-roll models predict a $n_T \approx 0$, i.e. a nearly scale-invariant spectrum.

Not only would the detection of $B-$modes be seen as evidence for inflation, but the amplitude of the signal directly measures the energy scale of inflation. Note that there is some confusion with regards to clearly detecting $B-$modes anisotropies generated by inflationary gravitational waves at recombination due to gravitational lensing. This may deform the polarization pattern on the sky relative to that at the last scattering surface, hence generating B type polarization even if only $E-$modes are intrinsically present at the last scattering surface [179].

Recall the ratio of tensor modes to scale power is

$$r \equiv \frac{\Delta_t^2(k)}{\Delta_s^2(k)} \tag{63}$$

where the amplitude of scalar fluctuations is measured to be $\Delta_s^2 \sim 10^{-9}$. Because $\Delta_t^2 \propto H^2 \propto V$, the tensor-to-scalar ratio r such that

$$V^{1/4} \sim \left(\frac{r}{0.01}\right)^{1/4} 10^{16} \;\; \text{GeV} \tag{64}$$

Scalar modes and tensor modes contain different information: features of scalar spectra are a result of the oscillation of matter-radiation plasma throughout the period up to recombination, thus encoding information abut the sound speed of the baryon-radiation plasma, the baryon fraction, and other cosmological constraints.

The tensor spectrum is determined by the wave motion of the evolving gravitational waves which primarily contain information about the expansion rate of the early Universe. Under standard assumptions, the amplitude of the GW background measures the expansion rate during Inflation. However, it has recently been pointed [63] out that there are mechanisms such that particles or strings act as sources for gravitational waves during inflation, and therefore the observation of a scale-invariant spectrum of gravity waves does not allow us to automatically derive the energy scale of inflation.

The current observations are in agreement with the generic predictions from inflation:

- **Flatness**

  For a universe composed of baryons, dark matter, photons, neutrinos, and dark energy,

  $$\Omega_{\text{tot}} = \Omega_b + \Omega_{\text{cdm}} + \Omega_\gamma + \Omega_\nu + \Omega_\Lambda \tag{65}$$

  inflation predicts $\Omega_{\text{tot}} = 1 \pm 10^{-5}$, whereas the data [15] predicts a value $\Omega_{\text{tot}} = 1 \pm 0.02$. Arguably, this prediction is not very robust, as inflation achieves a flat universe by design. However, one should remember that when Guth introduced inflation in 1980, the flatness of the universe was a non-trivial prediction.

- **Adiabatic Perturbations**

  The main components of the universe at the point of last scattering were baryons b, photons γ, and neutrinos ν, characterized by a respective energy density contrast δ_i.

  The adiabatic mode is such that the number of photons (or neutrinos, etc.) per baryon does not fluctuate, i.e. there is no variation in the relative density between different components

  $$\delta\rho_c = \delta\rho_b = \frac{3}{4}\delta\rho_\nu = \frac{3}{4}\delta\rho_\gamma \tag{66}$$

  Fluctuations of the inflaton field in single-field inflation shift the trajectory of the homogeneous background field, which affects the total density at different points of the universe but does not cause variations in the relative density between different components. Thus, single-field inflation is predicted to produce adiabatic primordial density perturbations such that all perturbations to the cosmological fluid (i.e. photons, neutrinos, cold dark matter particles, and baryons) originate from the same curvature perturbation $\mathcal{R}$.

  However, in inflationary models with more than one field, these perturbations may not necessarily be adiabatic. Even if the total density and spatial curvature is unperturbed, in the case of multifield inflation fluctuations orthogonal to the background trajectory affect the density between different components. Therefore, any violation of adiabaticity would be a signature for multifield inflation.

- **Scale-invariant spectrum**

  Recall that inflation produces a nearly constant or scale-invariant spectrum, i.e. the amplitude of a particular Fourier mode is drawn from a distribution with mean equal to zero and variance such that

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_{\mathcal{R}}(k) \tag{67}$$

  where $k^3 P_{\mathcal{R}}(k) \propto k^{n_s - 1}$ and $n_s \approx 1$. This is referred to as the Harrison-Zel'dovich-Peebles spectrum and a constant (i.e. no $k-$dependence) spectrum is predicted by most inflationary models. Measurements so far are in agreement with this generic prediction, though this alone isn't very robust, as we have expected such a spectrum without inflation.

  However, the implication of this is rather non-trivial, as this suggests that all Fourier modes have the same phase: Let's consider a Fourier mode with physical wavelength λ inside the horizon oscillating quantum mechanically with a frequency $1/\lambda$. However, before the end of inflation, the mode's wavelength is stretched greater than the Hubble radius $\lambda > H^{-1}$ such that no causal physics can alter them. After this, the amplitudes of the mode remains constant until upon re-entering the horizon; there causal physics again becomes relevant. Observe that since the fluctuation amplitude was constant outside the horizon, as the mode enters the horizon $\dot{\mathcal{R}}$ is small. Thinking of each Fourier mode as a linear combination of sine and cosine modes, only the cosine modes are excited by inflation (defining horizon re-entry at $t \equiv 0$).

  Using a simplistic model similar to simple harmonic motion to describe oscillations in the density field, the curvature perturbation $\mathcal{R}$ sources density fluctuations δ, evolving under gravity and pressure such that

$$\ddot{\delta} - c_s^2 \nabla^2 \delta = F_g[\mathcal{R}] \tag{68}$$

  where c_s^2 is the speed of sound and F_g is the gravitational source term. The CMB therefore provides a view of the condition of the density field at recombination, as matter density fluctuations were strongly coupled to radiation fluctuations in the plasma of the early universe.

This leads us to the key insight: We would expect that Fourier modes of the same phases and various wavelengths would coherently interfere, producing peaks and troughs (e.g. like a plucked guitar string) in the CMB power spectrum at the last-scattering surface—this is exactly what we see in the CMB spectrum. However, a guitar string has fixed nodes, producing a set of harmonics; there are no such restrictions for perturbations at the early universe. We would need a mechanism to produce coherent initial phases for all Fourier modes, which is exactly what inflation accomplishes: fluctuations freeze upon exiting the horizon, and therefore the phases for Fourier modes were set well before the modes of interest enter the horizon. This is why our previous implication was so powerful, as without such coherence such that phases were random (i.e. both sine and cosine modes were excited, as opposed to only cosine via inflation), the CMB power spectrum would be nearly flat (thus ruling out topological defects as the primary sources of structure formation).

(We should note that one objection to this reasoning in the 1990s was to simply write a theory of structure formation which obeys causality and produced only cosine modes [180,181]. This may seem logical at angular scale smaller than a degree $l > 200$; however, the negative cross-correlation $\langle TE \rangle$ between temperature and $E-$mode polarization on scales $100 < l < 200$ prompts us to disregard this possibility, because such scales were not within the horizon at recombination even though the signal is a result of phase coherence. This implies no causal physics could have produced such a result, leaving us this an inflationary scenario again.)

Appendix B

Background Perturbation Theory

This is to provide a brief (and *very* basic) background into cosmological perturbation theory, particularly aspects relevant to this dissertation. We recommend referring to the review *Cosmological perturbations* by Malik and Wands [182] and Dodelson [168] for more details.

Standard FLRW comsology is spatially homogeneous and isotropic. However, we need to describe the spatial inhomogeneity and anisotropy of the distribution of matter and energy in our observed universe (e.g. stars and galaxies forming clusters and superclusters of galaxies). As non-linear coupled second-order partial differential equations, Einstein's equations are notorious for lacking exact solutions which incorporate such traits. Therefore, cosmologists used perturbation theory: starting with a spatially homogenous and isotropic background, we can study increasing complex inhomogeneous perturbations order by order.

Using the symmetries of a flat FLRW background spacetime, we may decompose perturbations into independent scalar, vector, and tensor components, reducing Einstein's equations to a set of uncoupled ordinary differential equations.

The decomposition into scalar, vector, and tensor (SVT) perturbations is readily shown in Fourier space, where we define the Fourier components of a general perturbation $\delta Q(t, x)$ as

$$\delta Q(t, \mathbf{k}) = \int d^3x \delta Q(t, x) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{69}$$

We see that different Fourier modes (different wave numbers k) evolve independently as a consequence of translation invariance.

It can also be shown using the rotational invariance of the background that helicity scalar, vectors, and tensor evolve independently (see [182,12]).

Decomposing SVT perturbations into real space, it can be shown that:

- A 3-scalar corresponds to a helicity scalar

$$\alpha = \alpha^S \tag{70}$$

- A 3-vector β_i decomposes into a helicity scalar and vector

$$\beta_i = \beta_i^S + \beta_i^V \tag{71}$$

where

$$\beta_i^S = \nabla_i \hat{\beta}, \qquad \nabla^i \beta_i^V = 0 \tag{72}$$

- A 3-tensor decomposes into a helicity scalar, vector, and tensor. That is, a traceless, symmetric 3-tensor can be written as

$$\gamma_{ij} = \gamma_{ij}^S + \gamma_{ij}^V + \gamma_{ij}^T \tag{73}$$

where

$$\gamma_{ij}^S = \Big(\nabla_i \nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\Big)\hat{\gamma} \tag{74}$$

$$\gamma_{ij}^V = \frac{1}{2}(\nabla_i \hat{\gamma}_j + \nabla_j \hat{\gamma}_i), \quad \nabla_i \hat{\gamma}_i = 0 \tag{75}$$

$$\nabla_i \gamma_{ij}^T = 0 \tag{76}$$

For the purposes of this paper (and the sake of brevity), the details for metric perturbations are simply listed as follows (readers are encouraged to see Malik, Wands, Baumann, etc. for details):

- **Scalar Metric Perturbations**

Four scalar metric perturbations Φ, $B_{,i}$, $\Psi\delta_{ij}$ and $E_{,ij}$ may be constructed from

3-scalars, their derivatives and the background spatial metric

$$
\begin{aligned}
ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\
&= -(1+2\Phi)dt^2 + 2a(t)B_i dx^i dt + a^2(t)[(1-2\Psi)\delta_{ij} + 2E_{ij}]dx^i dx^j
\end{aligned} \tag{77}
$$

where we absorbed the $\nabla^2 E\delta_{ij}$ part of the helicity scalar E^S_{ij} in $\Psi\delta_{ij}$. Note that the SVT decomposition in real space is

$$B_i \equiv \partial_i B - S_i, \quad \text{where } \partial^i S_i = 0 \tag{78}$$

and

$$E_{ij} = 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}, \quad \text{where } \partial^i F_i = 0, \;\; h^i_i = \partial^i h_{ij} = 0 \tag{79}$$

(Recall that vector perturbations S_i and F_i are not produced during inflation, and also decay with the expansion of the universe).

The intrinsic Ricci scalar curvature of constant time hypersurfaces is

$$R_{(3)} = \frac{4}{a^2}\nabla^2\Psi \tag{80}$$

There are two scalar gauge transformations:

$$t \to t + \alpha \tag{81}$$

$$x^i \to x^i + \delta^{ij}\beta_{,j} \tag{82}$$

Under these coordinate transformations the scalar metric perturbations transform as:

$$\begin{aligned}
\Phi &\rightarrow \Phi - \dot{\alpha} && (83)\\
B &\rightarrow B + a^{-1}\alpha - a\dot{\beta} && (84)\\
E &\rightarrow E - \beta && (85)\\
\Psi &\rightarrow \Psi + H\alpha && (86)
\end{aligned}$$

Note tensor fluctuations are gauge-invariant. We also should note that Bardeen [183] introduced two gauge-invariant quantities (combinations of the scalar metric perturbations) which prove useful for extracting physical results:

$$\begin{aligned}
\Phi_B &\equiv \Phi - \frac{d}{dt}[a^2(\dot{E} - b/a)] && (87)\\
\Psi_B &\equiv \Psi + a^2 H(\dot{E} - B/a) && (88)
\end{aligned}$$

- **Scalar Matter Perturbations**

Matter perturbations are also gauge-dependent. For example, density and pressure perturbations transform under temporal gauge transformations as

$$\delta\rho \rightarrow \delta\rho - \dot{\bar{\rho}}\alpha, \qquad \delta p \rightarrow \delta p - \dot{\bar{p}}\alpha \tag{89}$$

However, we should note two gauge-invariant quantities formed from matter *and* metric perturbations:

First, the "curvature perturbation on uniform density hypersurfaces"

$$-\zeta \equiv \Psi + \frac{H}{\dot{\bar{\rho}}}\delta\rho \tag{90}$$

measures the spatial curvature of constant-density hypersurfaces, e.g. for adiabatic matter perturbations ζ remains constant outside the horizon.

Second, the "comoving curvature perturbation" $\mathcal{R}$, given by

$$\mathcal{R} = \Psi - \frac{H}{\bar{\rho} + \bar{p}} \delta q \tag{91}$$

measures the spatial curvature of comoving hypersurfaces. Note δq is the scalar part of the 3-momentum density $T_i^0 = \partial_i \delta q$.

It can be shown with the linearized Einstein equations and the Bardeen potential Ψ_B that

$$-\zeta = \mathcal{R} + \frac{k^2}{(aH)^2} \frac{2\bar{\rho}}{3(\bar{\rho} + \bar{p})} \Psi_B \tag{92}$$

Note that ζ and $\mathcal{R}$ are equal on superhorzion scales $k \ll aH$, where they become time-independent, and their correlations functions are equal at horizon crossing.

- **Vector Metric Perturbations**

  Vector type metric perturbations are defined

$$ds^2 = -dt^2 + 2a(t) S_i dx^i dt + a^2(t)[\delta_{ij} + 2F_{(i,j)}] dx^i dx^j \tag{93}$$

where $S_{i,i} = F_{i,i} = 0$. The vector gauge transformation is

$$x^i \ \rightarrow \ x^i + \beta^i, \quad \beta_{i,i} = 0 \tag{94}$$

They lead to the transformations

$$S_i \ \rightarrow \ S_i + a\dot{\beta}_i, \quad F_i \ \rightarrow \ F_i - \beta_i \tag{95}$$

- **Tensor Metric Perturbations**

  Tensor metric perturbations are defined as

$$ds^2 = -dt^2 + a^2(t)[\delta_{ij} + h_{ij}] dx^i dx^j \tag{96}$$

where $h_{ij,i} = h_i^i = 0$

Appendix C

A key (and perhaps most appealing) aspect of inflation is the mechanism by which quantum fluctuations of the inflaton source all structure in the universe and thus the primordial power spectra of scalar and tensor fluctuations $P_s(k)$ and $P_t(k)$. In this appendix, we detail how quantum physics gives rise to the macroscopic observables discussed in this paper.

Note that the amplitude of quantum fluctuations scales with the Hubble parameter H (i.e. to the de Sitter horizon, H^{-1} during inflation). Furthermore, during inflation, fluctuations are created at all length scales, or wavenumbers k. Created by quantum fluctuations via the familiar laws of quantum mechanics, they begin inside the horizon $k \gg aH$ (i.e. "subhorizon") and eventually exist as the comoving Hubble radius $(aH)^{-1}$ shrinks during inflation. These perturbations are referred to as "superhorizon" and as no causal physics can affect them, they are frozen until horizon re-entry at late times. All fluctuations must re-enter the horizon after inflation.

This is a crucial insight for observations: by characterizing inhomogeneity with comoving curvature fluctuations $\mathcal{R}$ and curvature perturbations on uniform density hypersurfaces ζ (see Appendix B), we realize that both $\mathcal{R}$ and ζ are constant on superhorizon scales such that their amplitude is not affected immediately after inflation (which is convenient, as we know virtually nothing about reheating). Thus, upon horizon re-entry, it is the constancy of $\mathcal{R}$ and ζ which allow us to connect fluctuations to observables (i.e. perturbations of the cosmic fluid measured by CMB anisotropies and LSS).

As the basis of this mechanism for generating inItial seeds of all structure is in fact quantum mechanics, cosmologists use the quantized simple harmonic oscillator, i.e. the time-dependent action

$$S = \int dt\Big(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2(t)x^2\Big) \equiv \int dtL \tag{97}$$

varied such that

$$\frac{\delta S}{\delta x} = 0 \implies \ddot{x} + \omega^2(t)x = 0 \tag{98}$$

with variables subsequently quantized to operators such that $[\hat{x}, \hat{p}] = i\hbar$, to guide them in computing quantum fluctuations in de Sitter space. Readers unfamiliar with this system are advertised to consult any beginning to advanced textbook on quantum theory.

Computing Quantum Fluctuations in de Sitter Space

Before computing inflationary fluctuations, recall we previously defined a gauge-invariant curvature perturbation $\mathcal{R}$ which we may compute at horizon exit (thus allowing us to ignore subhorizon physics and reheating until horizon re-entry of $\mathcal{R}$ modes). With the intuition of the quantized simple harmonic oscillator, we may use the SHO form to write the equation of motion for $\mathcal{R}$ and therefore study the quantization of scalar fluctuations during inflation.

We summarize the following procedure for purposes of clarity and guidance: Firstly, we expand the action for single-field slow-roll inflation to the second order in fluctuations in terms of $\mathcal{R}$. Secondly, we then derive the equation of motion for $\mathcal{R}$, resulting in the familiar SHO form. Next, we then approximate the solutions for this equation, as exact solutions are difficult to write. Fourthly, we will quantize the field $\mathcal{R}$, thus giving us boundary conditions on the mode functions. We may then fix these mode functions with the definition of a vacuum state, thus providing their large-scale limit. With this, we may finally compute the power spectra of curvature fluctuations at horizon crossing, thus providing cosmologists a connection between quantum fluctuations and cosmic observables (e.g. CMB anisotropies).

We will begin with the case of scalar perturbations, and subsequently generalize to the case of tensor perturbations.

(Note that the following closely the derivations of Maldacena [99] and Baumann[12]).

Scalar Perturbations

We begin with the familiar action of single-field slow-roll models

$$S = \frac{1}{2} \int d^4x \sqrt{-g}[R - (\nabla\phi)^2 - 2V(\phi)] \tag{99}$$

Our first step is to expand this action to the second order in $\mathcal{R}$. We will choose the following gauge for dynamical fields g_{ij} and ϕ

$$\delta\phi = 0, \qquad g_{ij} = a^2[(a - 2\mathcal{R})\delta_{ij} + h_{ij}], \qquad \partial_i h_{ij} = h_i^i = 0 \tag{100}$$

to fix time and spatial reparametrizations such that the inflaton field is unperturbed and all scalar degrees of freedom of parametrized by the metric fluctuations $\mathcal{R}(t, \mathbf{x})$, where recall $\mathcal{R}$ measure the spatial curvature of constant-ϕ hypersurfaces, $R^{(3)} = 4\nabla^2\mathcal{R}/a^2$. This is crucial: remember, because $\mathcal{R}$ remains constant outside the horizon, we may solely compute correlation functions of $\mathcal{R}$ at horizon crossing.

This following procedure is rather computationally detailed, but we will walk through the major steps:

Consider the slow-roll background

$$\mathrm{d}^2 s = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j = a(\tau)^2(\mathrm{d}\tau^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j) \tag{101}$$

using the scale factor $a(t)$ and Hubble parameter $H(t) \equiv \partial_t \ln a$.

In order to study fluctuations in the metric, we introduce ADM formalism (see [184] for further details). In ADM formalism, we formulate General Relativity such that spacetime is foliated into three-dimensional hypersurfaces

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + g_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t) \tag{102}$$

where g_{ij} is the three-dimensional metric on slice of constant t, $N(\mathbf{x})$ is the lapse function, and $N_i(\mathbf{x})$ is the shift function. Note that metric perturbations Φ and B are related to $\mathcal{R}$ by this formalism (i.e. represented by $N(\mathbf{x})$ and $N_i(\mathbf{x})$, except the latter were chosen to be non-dynamical Lagrangian multipliers in the action, see [184]). With this formalism, we can then write the action S of eq. (100) as

$$\begin{aligned} S = \frac{1}{2}\int d^4x\sqrt{-g}\Big[&NR^{(3)} - 2NF + N^{-1}(E_{ij}E^{ij} - E^2) + \\ &N^{-1}(\dot{\phi} - N^i\partial_i\phi)^2 - Ng^{ij}\partial_i\phi\partial_j\phi - 2V\Big] \end{aligned} \tag{103}$$

where

$$E_{ij} \equiv \frac{1}{2}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i), \qquad E = E_i^i \tag{104}$$

E_{ij} is related to the extrinsic curvature of the three-dimensional spatial slices $K_{ij} = N^{-1}E_{ij}$. It can be shown that constraint equations for the Lagrange multipliers N and N^i for this action are given by

$$\nabla_i[N^{-1}(E^i_j - \delta^i_j E)] = 0, \tag{105}$$
$$R^{(3)} - 2V - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi}^2 = 0 \tag{106}$$

We next define the shift vector N_i into irrotational (scalar) and incompressible (vector) parts

$$N_i \equiv \psi_{,i} + \tilde{N}_i, \quad \text{where} \quad \tilde{N}_{i,i} = 0 \tag{107}$$

and the lapse function as

$$N \equiv 1 + \alpha \tag{108}$$

With this, we may solve the constraint equations above. It can be shown that eq. (106) at first order implies

$$\psi_1 = -\frac{\mathcal{R}}{H} + \frac{a^2}{H}\epsilon_V \partial^{-2}\dot{\mathcal{R}} \tag{109}$$

and eq. (107) implies

$$\alpha_1 = \frac{\dot{\mathcal{R}}}{H}, \quad \partial^2 \tilde{N}_i^{(1)} = 0 \tag{110}$$

where ∂^{-2} is defined such that $\partial^{-2}(\partial^2\phi) = \phi$.

We then substitute the first-order solutions for N and N_i into the action eq. (104) resulting in the second-order action

$$S = \frac{1}{2}\int d^4x a^3 \frac{\dot{\phi}^2}{H^2}\left[\dot{\mathcal{R}}^2 - a^{-2}(\partial_i\mathcal{R})^2\right] \tag{111}$$

We proceed by defining the "Mukhanov variable"

$$v \equiv z\mathcal{R}, \quad \text{where} \quad z^2 \equiv z^2\frac{\dot{\phi}^2}{H^2} = 2a^2\epsilon \tag{112}$$

and using conformal time τ gives us an action for a canonically normalized scalar, the “Mukhanov action”

$$S_{(2)} = \frac{1}{2}\int d\tau d^3x\Big[(v')^2 + (\partial_i v)^2 + \frac{z''}{z}v^2\Big] \tag{113}$$

where we've differentiated with respect to τ.

And with our convention of the Fourier expansion of the field v, we arrive at the Mukhanov Equation

$$v_k'' + \Big(k^2 - \frac{z''}{z}\Big)v_k = 0 \tag{114}$$

which is exact in linear theory. Note for large (i.e. subhorizon) k, this looks exactly like a free oscillator equation which we can quantize! However, the Mukhanov Equation is hard to solved in full generality because z depends on background dynamics. We hence discuss approximate solutions in the pure de Sitter case to gain some intuition of the solutions:

As in the familiar case of the quantized SHO, we promote the field v and conjugate momentum v' as quantum operators

$$v \to \hat{v} = \int \frac{d\mathbf{k}^3}{(2\pi)^3}\Big[v_k(\tau)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + v_k^*(\tau)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\Big] \tag{115}$$

or alternatively expressed in Fourier decomposition with components $v_{\mathbf{k}}$

$$v_{\mathbf{k}} \to \hat{v}_{\mathbf{k}} = v_k(\tau)\hat{a}_{\mathbf{k}} + v_{-k}^*(\tau)\hat{a}_{-\mathbf{k}}^\dagger \tag{116}$$

where the creation and annihilation operators $\hat{a}_{-\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ satisfy the commutation relation $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^\dagger] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$ if and only if the mode functions are normalized such as

$$\langle v_k, v_k\rangle \equiv \frac{i}{\hbar}(v_k^* v_k' - v_k'^* v_k) = 1 \tag{117}$$

This is in fact the first boundary condition on solutions of the Mukhanov Equation. Now that we've established the quantum normalization, the second boundary condition is chosen to fix the mode functions via a choice of vacuum. This vacuum state for fluctuations

$$\hat{a}_{\mathbf{k}}|0\rangle = 0, \tag{118}$$

is the so-called “Bunch-Davies vacuum” (i.e. the Minkowski vacuum of comoving observer in the far past when all comoving scales were far inside the Hubble Horizon, $\tau \to -\infty$ or $k \ggg aH$). In this subhorizion limit, we find a SHO equation with time-independent frequency such that

$$v_k'' + k^2 v_k = 0 \tag{119}$$

As in introductory quantum mechanics, by requiring the vacuum to be the minimum energy state, we find a unique solution. Therefore, by imposing the initial condition

$$\lim_{\tau \to -\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{120}$$

along with the previous boundary conditions, the mode functions are fixed on all scales.

Now let's consider the pure de Sitter space limit $\epsilon \to 0$ where H is constant, and

$$\frac{z''}{z} = \frac{a''}{a} = \frac{2}{\tau^2} \tag{121}$$

In such a de Sitter background, we are given a mode equation

$$v_k'' + \Big(k^2 - \frac{2}{\tau^2}\Big) v_k = 0 \tag{122}$$

with solutions

$$v_k = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \Big(1 - \frac{i}{k\tau}\Big) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \Big(1 + \frac{i}{k\tau}\Big) \tag{123}$$

By fixing $\alpha = 1, \beta = 0$ with the subhorizon limit $|k\tau| \gg 1$, this gives us a unique Bunch-Davies mode function

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \Big(1 - \frac{i}{k\tau}\Big) \tag{124}$$

Power Spectrum

Now let's compute the power spectrum in quasi-de Sitter space:

By defining a field $\hat{\psi}_{\mathbf{k}} \equiv \hat{v}_{\mathbf{k}}/a$, we compute the power spectrum

$$
\begin{aligned}
\langle \hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}'_{\mathbf{k}}(\tau)\rangle &= (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{|v_k(\tau)|^2}{a^2} \\
&= (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H^2}{2k^3}(1+k^2\tau^2)
\end{aligned}
\tag{125}
$$

Note that by taking the superhorizon limit $|k\tau| << 1$, this approaches a constant

$$
\langle \hat{\psi}_{\mathbf{k}}(\tau)\hat{\psi}'_{\mathbf{k}}(\tau)\rangle \to (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H^2}{2k^3}, \quad \text{or} \quad \Delta^2_{\psi} = \left(\frac{H}{2\pi}\right)^2 \tag{126}
$$

Observe that by using $\psi = v/a$ in the de Sitter limit, we may now compute the power spectrum of $\mathcal{R} = \frac{H}{\dot{\phi}}\psi$ at horizon crossing $a(t_*)H(t_*) = k$, giving

$$
\langle \mathcal{R}_{\mathbf{k}}(t)\mathcal{R}_{\mathbf{k}'}(t)\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\frac{H_*^2}{2k^3}\frac{H_*^2}{\dot{\phi}^2} \tag{127}
$$

where t_*,H_*, etc. denoted evaluation at horizon crossing. This allows us to define the dimensionless power spectrum $\Delta^2_{\mathcal{R}}(k)$ by

$$
\langle \mathcal{R}_{\mathbf{k}}\mathcal{R}_{\mathbf{k}'}\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')P_{\mathcal{R}}(k), \qquad \Delta^2_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2}P_{\mathcal{R}}(k) \tag{128}
$$

where the real space variance of $\mathcal{R}$ is $\langle \mathcal{R}\mathcal{R}\rangle = \int_0^\infty \Delta^2_{\mathcal{R}}(k) d\ln k$. This results in

$$
\Delta^2_{\mathcal{R}}(k) = \frac{H_*^2}{2k^3}\frac{H_*^2}{\dot{\phi}_*^2} \tag{129}
$$

Again, we stress that because $\mathcal{R}$ approaches a constant on super-horizon scales the spectrum at horizon crossing determines the future spectrum until a given fluctuation mode re-enters the horizon. We also note that when horizon crossing $a_*H_* = k$ has different values, different modes will exist the horizon at slightly different times.

Observe that our result is valid at a slowly time-evolving quasi-de Sitter space as we computed the power spectrum at a specific instant, i.e. horizon crossing. Therefore, we have calculated the power spectrum for single-field slow-roll inflation. (The Mukhanov Equation usually must be solved numerically for non-slow-roll cases, as the background evolution must be tracked in a more precise manner.)

Tensor Perturbations

Now let's repeat this computation for tensor perturbations, as quantum fluctuations during inflation also excite tensor metric perturbations h_{ij}. Although the computation itself differs, the calculation is quite similar to the scalar case previously detailed:

By expanding the Einstein-Hilbert action, we can compute the second-order action for tensor fluctuations

$$S_{(2)} = \frac{M_{\rm Pl}^2}{8} \int \mathrm{d}\tau \mathrm{d}x^3 a^2 [(h'_{ih})^2 - (\partial_t h_{ij})^2] \tag{130}$$

where factors of $M_{\rm Pl}$ are included to emphasize h_{ij} is dimensionless.

Defining the following Fourier expansion to be

$$h_{ij} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{s=+,\times} \epsilon_{ij}^s(k) h_{\mathbf{k}}^s(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{131}$$

where $\epsilon_{ii} = k^i \epsilon_{ij} = 0$ and $\epsilon_{ij}^s(k)\epsilon_{ij}^{s'}(k) = 2\delta_{ss'}$, the tensor action $S_{(2)}$ becomes

$$S_{(2)} = \sum_s \int \mathrm{d}\tau \mathrm{d}\mathbf{k} \frac{a^2}{4} M_{\rm Pl}^2 [h_{\mathbf{k}}^{s'} h_{\mathbf{k}}^{s'} - k^2 h_{\mathbf{k}}^s h_{\mathbf{k}}^s] \tag{132}$$

with the polarization of a gravitational wave denoted by $h_{\mathbf{k}}^s$.

We then define the canonically normalized field

$$v_{\mathbf{k}}^2 \equiv \frac{a}{2} M_{\rm Pl} h_{\mathbf{k}}^s, \tag{133}$$

where

$$\frac{a''}{a} = \frac{2}{\tau^2} \tag{134}$$

holds in de Sitter space, resulting in

$$S_{(2)} = \sum_s \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}^3\mathbf{k} \Big[(v_{\mathbf{k}}^{s'})^2 - \Big(k^2 - \frac{a''}{a}\Big)(v_{\mathbf{k}}^s)^2 \Big] \tag{135}$$

Notice this is another copy of the free field action eq. (114).

Note that we can now write the polarization of a gravitational wave as a renormalized massless field in de Sitter space

$$h^s_{\mathbf{k}} = \frac{2}{M_{pl}}\psi^2_{\mathbf{k}}, \qquad \psi^s_{\mathbf{k}} \equiv \frac{v_{\mathbf{k}}}{a} \tag{136}$$

where $\psi \equiv v/a$ is used as before. This allows us to simply write down the answer for Δ^2_h, the power spectrum for a single polarization of tensor perturbations with the correct normalization factor

$$\Delta^2_h = \frac{4}{M^2_{\mathrm{Pl}}}\Big(\frac{H_*}{2\pi}\Big)^2 \tag{137}$$

Therefore, the dimensionless power spectrum of tensor fluctuations is given as

$$\Delta^2_t = 2\Delta^2_h(k) = \frac{2}{\pi^2}\frac{H^2_*}{M^2_{\mathrm{Pl}}} \tag{138}$$

Energy Scale of Inflation

We may now define the tensor-to-scalar ratio r as

$$r \equiv \frac{\Delta^2_t(k)}{\Delta^2_s(k)} \tag{139}$$

Under normal assumptions, we may use this ratio to extrapolate the energy scale of inflation: Because the amplitude of scalar fluctuations Δ^2_s is fixed such that $\Delta^2_s \equiv \Delta^2_{\mathcal{R}} \approx 10^{-9}$ and the amplitude of tensor fluctuations $\Delta^2_t(k)$ is directly proportaional to $H^2 \approx V$, we find the relation between the energy scale of inflation $V^{1/4}$ and $r_* \equiv r(\phi_{\mathrm{CMB}})$

$$V^{1/4} \approx \big(\frac{r}{0.01}\big)^{1/4} \times 10^{16} \ \ \mathrm{GeV} \tag{140}$$

where $r \geq 0.01$ would correspond to any inflationary energy at Grand Unified Theory (GUT) scales, a result which would be the first direct insight into physics at such a scale.

The Lyth Bound

The tensor-to-scalar ratio may be related to the evolution of the inflaton field such that [185]

$$r(N) = \frac{8}{M_{\text{Pl}}^2}\big(\frac{d\phi}{dN}\big)^2 \tag{141}$$

It can then be calculated [186] that between the time when CMB fluctuations exited the horizon at N_{CMB} and the end of inflation at N_{end}, the total field evolution is

$$\frac{\Delta\phi}{M_{\text{Pl}}} = \int_{N_{\text{end}}}^{N_{\text{CMB}}} \sqrt{\frac{r}{8}}\text{d}N \tag{142}$$

Observe the evolution of r is highly constrained by slow-roll parameters. Taking the lower bounds of slow-roll models, we obtain the estimate

$$\frac{\Delta\phi}{M_{\text{Pl}}} = \mathcal{O}(1) \times \big(\frac{r}{0.01}\big)^{1/2} \tag{143}$$

A detection of a large value of r (i.e. $r > 0.01$) would thus not only signify a high-scale of inflationary energy, but also correlate with $\Delta\phi > M_{\text{Pl}}$ (i.e. super-Planckian field evolution) and therefore be considered a signature of large-field inflation.

Thus, we have successfully calculated the power spectra of inflationary scalar and tensor perturbations, written as

$$\Delta_s^2(k) \equiv \Delta_{\mathcal{R}}^2(k) = \frac{1}{8\pi^2}\frac{H^2}{M_{\text{Pl}}^2}\frac{1}{\epsilon}\Big|_{k=aH} \tag{144}$$

$$\Delta_t^2(k) \equiv 2\Delta_h^2(k) = \frac{2}{\pi^2}\frac{H^2}{M_{\text{Pl}}^2}\Big|_{k=aH} \tag{145}$$

where

$$\epsilon = -\frac{d\ln H}{dN} \tag{146}$$

resulting in a tensor-to-scalar ratio at the time of horizon crossing $k = a(t_*)H(t_*)$ to be

$$r \equiv \frac{\Delta_t^2}{\Delta_r^2} = 16\epsilon_* \tag{147}$$

We now continue by calculating the scale dependence of the spectra and the time-dependence of the Hubble parameter:

$$n_s - 1 \equiv \frac{d\ln\Delta_s^2}{d\ln k}, \qquad n_t \equiv \frac{d\ln\Delta_t^2}{d\ln k} \tag{148}$$

Using n_s and without loss of generality, we split n_s into two factors such that

$$\frac{d\ln\Delta_s^2}{d\ln k} = \frac{d\ln\Delta_s^2}{dN} \times \frac{dN}{d\ln k} \tag{149}$$

The first factor is the derivative with respect to e-folds N such that

$$\frac{d\ln\Delta_s^2}{dN} = 2\frac{d\ln H}{dN} - \frac{d\ln\epsilon}{dN} \tag{150}$$

where the first term is simply -2ϵ and the second is shown to be [12]

$$\frac{d\ln\epsilon}{dN} = 2(\epsilon - \eta), \quad \text{where} \quad \eta = -\frac{d\ln H_{,\phi}}{dN} \tag{151}$$

For the second factor $dN/d\ln k$ evaluated at the horizon crossing, we calculate

$$\frac{dN}{d\ln k} = \left[\frac{d\ln k}{dN}\right]^{-1} = \left[1 + \frac{d\ln H}{dN}\right]^{-1} \approx 1 + \epsilon \tag{152}$$

Therefore, we may conclude to the first order that the slow-roll Hubble parameters are given by

$$n_s - 1 = 2\eta_* - 4\epsilon_* \tag{153}$$

and

$$n_t = -2\epsilon_* \tag{154}$$

As discussed throughout this paper, observational cosmology has long had the goal of measuring n_s and n_t (particularly deviation from exact scale-invariance $n_s = 1$ and $n_t = 0$) as probe into inflationary dynamics (e.g. slow-roll parameter ϵ and η).

Slow-Roll Primordial Constraints

We are now in a position to summarize constraints of the single-field slow-roll inflation. Recall that measurements of scalar and tensor power spectra directly relate to information concerning the slow-roll parameter and shape of inflationary potential. In this limit, the Hubble parameter and potential slow-roll parameters are related as

$$\epsilon \approx \epsilon_V, \qquad \eta \approx \eta_V - \epsilon_V \tag{155}$$

Expressing the scalar and tensor spectra in terms of $V(\phi)$ and ϵ_V (or $V_{,\phi}$), we find

$$\Delta_s^2(k) \approx \frac{1}{24\pi^2}\frac{V}{M_{\rm Pl}^4}\frac{1}{\epsilon_V}\bigg|_{k=aH}, \qquad \Delta_t^2(k) \approx \frac{2}{3\pi^2}\frac{V}{M_{\rm Pl}^4}\bigg|_{k=aH} \tag{156}$$

The scalar and tensor scale dependence are therefore

$$n_s - 1 \equiv \frac{d\ln\Delta_s^2}{d\ln k} \approx 2\eta_V^* - 6\epsilon_V^* \tag{157}$$

$$n_t \equiv \frac{d\ln\Delta_t^2}{d\ln k} = -2\epsilon_V^* \tag{158}$$

and the tensor-to-scalar ratio is

$$r \equiv \frac{\Delta_t^2}{\Delta_s^2} = 16\epsilon_V^*. \tag{159}$$

Notice how we may therefore conclude the consistency condition between the tensor-to-scalar ratio r and the tensor tilt n_t

$$r = -8n_t \tag{160}$$

Appendix D

Our aim in this appendix is to provide a background of the "In-In" formalism to compute correlation functions in a time-dependent background [187-192].

First, let's recall how we analysis correlation functions in Quantum Field Theory as applied to particle physics. Here, we impose asymptotic conditions at very early times $-\infty$ and very late times $+\infty$ as these states may be assumed to be non-interacting at $\pm\infty$ in Minkowski space. That is to say, the asymptotic state is set as the vacuum state of the free Hamiltonian H_0. So, we take some S-matrix of the probability for a state $|\Psi\rangle$ to transition to state $|\Psi'\rangle$ at the far future $+\infty$, such that

$$\langle \Psi' \mid S \mid \Psi \rangle = \langle \Psi'(+\infty) \mid \Psi(-\infty) \rangle. \tag{161}$$

Calculating cosmological correlations functions differs from the familiar technique in quantum field theory in several respects, as we are interested in evaluating expectation values of products of fields at a fixed time. In our case, we impose conditions not at $\pm\infty$ but only in the limit of early times, when the wavelength is deep inside the horizon, i.e. superhorizon primordial perturbations generated during inflation. Note that fields in this scenario are fluctuations of the scalars and metric, and their conjugate momenta. In this limit, these fields in the interaction picture have the same form as in Minkowski space, prompting the definition of the free vacuum in Minkowski space, the "Bunch-Davies vacuum."

Deriving the $\mid \text{in}\rangle$ Vacuum in the Interaction Picture

First, we split the Hamiltonian into free and interacting parts

$$H = H_0 + H_{\text{int}} \tag{162}$$

such that the free-field H_0 is quadratic in perturbations. In order to describe the time-evolution of cosmological perturbations to study non-Gaussianities, we must study higher-order correlations in the interaction Hamiltonian H_{int}. We can define the evolution of states in H_{int} with the time-evolution operator written as

$$U(\tau_2, \tau_1) = T \exp\Big(-i \int_{\tau_1}^{\tau_2} \mathrm{d}\tau' H_{\text{int}}(\tau') \Big) \tag{163}$$

where T is the time-ordering operator. $U(\tau_2, \tau_1)$ relates the interacting vacuum at soe time $|\Omega(t)\rangle$ to the free Bunch-Davies vacuum $|0\rangle$. To do this, we begin by expanding in $\Omega(\tau)$ in eigenstates of the free Hamiltonian

$$\mid \Omega\rangle = \sum_n \mid n\rangle\langle n \mid \Omega(\tau)\rangle \tag{164}$$

and then evolve $\mid \Omega(\tau)\rangle$ as

$$\mid \Omega(\tau_2)\rangle = U(\tau_2, \tau_1) \mid \Omega(\tau_1)\rangle = \mid 0\rangle\langle 0 \mid \Omega\rangle + \sum_{n\geq 1} e^{+iE_n(\tau_2-\tau_1)} \mid n\rangle\langle n \mid \Omega(\tau_1)\rangle \tag{165}$$

Here $\tau_2 = -\infty(1 - i\epsilon)$ projects out all excited states. Therefore, for the interacting vacuum $\tau = -\infty(1 - i\epsilon)$ and the free vacuum $\mid 0\rangle$, this implies

$$\Omega(-\infty(1 - i\epsilon)\rangle = \mid 0\rangle\langle 0 \mid \Omega\rangle \tag{166}$$

The interacting vacuum at some time τ may then be written as

$$\begin{aligned} \mid \text{in}\rangle \equiv \mid \Omega(\tau)\rangle &= U(\tau, -\infty(1 - i\epsilon) \mid \Omega(-\infty(1 - i\epsilon))\rangle & (167) \\ &= T \exp\Big(-i \int_{-\infty(1-i\epsilon)}^{\tau} \mathrm{d}\tau' H_{\text{int}}(\tau') \Big) \mid 0\rangle\langle 0 \mid \Omega\rangle & (168) \end{aligned}$$

Expectation Values

Now we are able to compute expectation values in this formalism. For an expectation value $\langle W(\tau)\rangle$ of a product of operators $W(\tau)$ at time τ, we find

$$\begin{aligned}\langle W(\tau)\rangle &\equiv \frac{\langle \text{in} \mid W(\tau) \mid \text{in}\rangle}{\langle \text{in} \mid \text{in}\rangle} \\ &= \left\langle 0\middle|\left(T\exp(-i\int_{-\infty^+}^{\tau} H_{\text{int}}(\tau')d\tau')\right)^{\dagger} W(\tau)\ \left(T\exp(-i\int_{-\infty^+}^{\tau} H_{\text{int}}(\tau'')d\tau'')\right)\middle|0\right\rangle\end{aligned} \tag{169}$$

or using the notation $-\infty^{\pm} \equiv -\infty(1 \mp i\epsilon)$ and anti-time ordering operator $\bar{T}$,

$$\langle W(\tau)\rangle = \left\langle 0\middle|\left(\bar{T}e^{-i\int_{-\infty^-}^{\tau} H_{\text{int}}(\tau')d\tau'}\right)W(\tau)\left(Te^{-i\int_{-\infty^+}^{\tau} H_{\text{int}}(\tau'')d\tau''}\right)\middle|0\right\rangle \tag{170}$$

We can now evaluate $\langle W(\tau)\rangle$ perturbatively in the interaction Hamiltonian H_{int} to compute higher-order correlation functions.

For example, we may use this formalism to compute three-point correlation function for inflationary models, written as

$$\begin{aligned}&\langle \mathcal{R}_{\mathbf{k}_1}\mathcal{R}_{\mathbf{k}_2}\mathcal{R}_{\mathbf{k}_3}\rangle(\tau) = \\ &\left\langle 0\middle|\left(\bar{T}e^{-i\int_{-\infty^-}^{\tau} H_{\text{int}}(\tau')d\tau'}\right)\mathcal{R}_{\mathbf{k}_1}(\tau)\mathcal{R}_{\mathbf{k}_2}(\tau)\mathcal{R}_{\mathbf{k}_3}(\tau)\left(Te^{-i\int_{-\infty^+}^{\tau} H_{\text{int}}(\tau'')d\tau''}\right)\middle|0\right\rangle\end{aligned} \tag{171}$$

Perturbative Expansion of Correlation Functions

Recall that by eq. (116) in Appendix C, we may promote the Mukhanov variable $v = 2a^2\epsilon\mathcal{R}$ to an operator and expand in terms of creation and annihilation operators

$$v_{\mathbf{k}} \to \hat{v}_{\mathbf{k}} = v_k(\tau)\hat{a}_{\mathbf{k}} + v^*_{-k}(\tau)\hat{a}^{\dagger}_{-\mathbf{k}} \tag{172}$$

The mode functions $v_k(\tau)$ were defined uniquely by initial state boundary conditions when all modes were deep inside the horizon

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\Big(1 - \frac{i}{k\tau}\Big) \tag{173}$$

The free two-point correlation function is

$$\langle 0 \mid \hat{v_{\mathbf{k}_1}}(\tau_1)\hat{v_{\mathbf{k}_2}}(\tau_2) \mid 0\rangle = (2\pi)^3\delta(\mathbf{k}_1 + \mathbf{k}_2)G_{k_1}(\tau_1, \tau_2) \tag{174}$$

with

$$G_{k_1}(\tau_1, \tau_2) \equiv v_k(\tau_1)v_k^*(\tau_2). \tag{175}$$

Expanding eq. (173) in powers of H_{int} and denoting $W(\tau) = \mathcal{R}_{\mathbf{k}_1}(\tau)\mathcal{R}_{\mathbf{k}_2}(\tau)\mathcal{R}_{\mathbf{k}_3}(\tau)$, we find:

- **at zeroth order**

$$\langle W(\tau)\rangle^{(0)} = \langle 0 \mid W(\tau) \mid 0\rangle \tag{176}$$

where we note this term naturally vanishes for exact Gaussian conditions in bispectrum calculations

- **at first order**

$$\langle W(\tau)\rangle^{(1)} = 2\,\mathrm{Re}\Big[-i\int_{-\infty^+}^{\tau} d\tau' \langle 0 \mid W(\tau)H_{\text{int}}(\tau') \mid 0\rangle\Big] \tag{177}$$

where this consequently is the leading term in bispectrum calculations

- **at second order**

$$\begin{aligned}\langle W(\tau)\rangle^{(2)} = -2\,\mathrm{Re}\Big[&\int_{-\infty^+}^{\tau} d\tau' \int_{-\infty^+}^{\tau'} d\tau'' \langle 0 \mid W(\tau)H_{\text{int}}(\tau')H_{\text{int}}(\tau'') \mid 0\rangle \\ &+ \int_{-\infty^-}^{\tau} d\tau' \int_{-\infty^+}^{\tau} d\tau'' \langle 0 \mid H_{\text{int}}(\tau')W(\tau)H_{\text{int}}(\tau'') \mid 0\rangle\Big]\end{aligned} \tag{178}$$

where we have used Wick's theorem to express the result in terms of two-point functions.

Appendix E

In this section we detail some of the theoretical concerns and mysteries which still plague the inflationary paradigm and inflationary model building. Furthermore, we discuss alternatives to inflation and the possibility to observationally distinguish or validate these models.

Issues with Inflationary Model Building

In spite of the large varieties of inflationary models presented these past three decades, theorists have found that in actuality, constructing a self-consistent and explicit model is a delicate task. We present two such challenges [104]:

- *The Eta Problem*

  For slow-roll models, the mass of the inflaton field must be light enough, i.e. $m \ll H$, to maintain a flat potential $V(\phi)$ and yet the natural mass of a light particle of order H in the inflationary background. Such a result would impair slow-roll inflation itself, as such inflaton masses generate contributions to the eta slow-roll parameter. We are essentially confronted with a fine-tuning problem of η, i.e. "Why is inflation so light?". As the need for an inflationary universe is arguably simply a fine-tuning issue to begin with, this implication is particularly theoretically distressful for our simplest case of inflation [193].

- *Variation of the Potential*

  Large-field potentials take the following general form

$$V(\phi) = \sum_{n=0}^{\infty} \lambda_n m_{\text{fund.}}^{4-n} \phi^n \tag{179}$$

  where $m_{\text{fund.}}$ represents typical scales in the theory and λ_n's are dimensionless couplings of order $\mathcal{O}(1)$. In order for the field theory description to hold, $m_{\text{fund.}}$ must be greater than M_{Pl}. This also presents a fine-tuning problem, as the shape of the potential varies over a scale of order $m_{\text{fund.}} \ll M_{\text{Pl}}$ [194].

Issues with Inflationary Paradigm

With these two model-dependent concerns regarding inflation, we may now turn to some of the more conceptual problems and ambiguities which have agonized theorists over the years. Here is a sampling:

- **Fine-Tuning Problems**

  The initial conditions for inflation to start with are poorly understood. For instance, note that the simple slow-roll analysis of inflation we have relied upon assumes small initial inflaton velocities, and initial homogeneities in the inflaton which are not large enough to prevent inflation. Questions concerning these and other initial conditions of inflation have concerned the field since inception. For instance, one such approach to study the initial state of the Universe is "Quantum Cosmology," which discusses the possible "probability" of inflation and the boundary conditions from which it is most likely to occur [195-198]. (See [199] for a pedagogical introduction to this field.)

  For the inflationary paradigm itself, however, the question concerning the initial conditions manifest in fine-tuning problems. Those detailed below are particularly worrisome for small-field models $\Delta\phi \ll M_{\text{Pl}}$:

  1. *The Overshoot Problem*

     One manifestation of this lack of understanding is the so-called "overshoot problem" first discussed in [200]. If the initial velocity of the inflaton near the inflationary potential is non-negligible or if the inflaton begins at a modest distance uphill from the inflationary potential, it *overshoots* that region without sourcing accelerated expansion. For small-field models in particular, the Hubble friction is not efficient enough to slow the field before it reaches the region of interest.

  2. *The Patch Problem*

     The inflaton field must be smooth over a few times the horizon size at that time to start inflation. However, initial inhomogeneites in the inflaton field provide a gradient energy that also could hinder accelerate expansion [12].

3. *"Fine-Tuned" Potential Problem*

   For single-field models to transition from $\epsilon = 1$ at $\phi_{\rm end}$ to $\epsilon \ll 1$ at $\phi_{\rm CMB}$, i.e. $r = 16\epsilon \ll 1$ to $\epsilon = 1$, within 60 $e-$folds in fact would require highly fine-tuned potentials [201].

- **Reheating**

  Our knowledge of reheating is very limited, as the subject is composed of various theoretical difficulties. How did inflation end? How did it result in a universe filled much radiation and elementary particles of the Standard Model? For more information concerning issues of reheating and preheating, see [7].

- **Eternal Inflation and the Measure Problem**

  The idea behind eternal inflation is that although inflation ends locally to produce pockets of FLRW universes, regions exist where quantum fluctuations keep the field at high values of potential energy. Those regions keep expanding exponentially and produce more volume of inflationary regions.

  This leaves us with the measure problem: the likelihood of the initial conditions of inflation and dependency of inflationary predictions is based on the relative probabilities of the inflationary and non-inflationary patches of the universe or multiverse. Furthermore, arguably all fine-tuning problems we previously discussed are manifestations of the measure problem.

  For more discussion of this topic, see [202-203].

Alternatives to Inflation

Naturally there could be an altogether different paradigm which resolves the various problems with Big Bang cosmology. However, we should note that in some sense, most of these models suffer from similar problems as inflation, particularly with regards invoking new physics not understood. One such leading alternative is the "Ekpyrotic (or Cyclic) Cosmology" model [64,204].

As opposed to the inflationary short burst of accelerated expansion from some energetic initial state, the ekyprotic universe slowly contracts from a cold beginning, then subsequently relies on a bounce (i.e. the contracting phase to smoothly connect with the Big Bang expansion) which leads to the standard decelerating, expanding FLRW cosmology. Thus, these models solve the various problems of standard Big Bang cosmology. The cyclic model furthermore posits the ekpyrotic phase occurs at an infinite number of times.

Theoretically, the viability of this model depends on whether a bounce could possibility happen or not. For there to have actually been a bounce would require a violation of the null energy condition (NEC) (see Chapter 4) such that

$$2M_{\rm Pl}^2\dot{H} = -(\rho + p) > 0 \tag{180}$$

A different approach is provided by variable speed of light (VSL) models [66,205,206]. Instead of modifying the matter content of the Universe such that Einsteinian gravity becomes repulsive via inflationary expansion, VSL models modify the local speed of light to resolve the various puzzles with the Standard Big Bang model. Thus, in the very early universe such distant regions in the expanding universe would have been in causal contact. Likewise, in relation to our discussion above, such theories may be described by their speed of sound: an extraordinarily large sound speed in the very early universe would result in a sound horizon far greater than the comoving Hubble radius, therefore solving the Horizon Problem. Such a sound speed decays with time, causing the comoving sound horizon to shrink. Indeed, such a mechanism utilizing a large, decaying sound speed in the early Universe has been proposed to set scale-invariant density fluctuations [105-107]. Refer to [206] for more discussion of the VSL theories and their various manifestations (e.g. soft-breaking of Lorentz invariance, hard-breaking, bimetric theories. etc.), and their testability.

Note that in both the cases of ekpyrosis and VSL, a generic phenomenological prediction is the lack of significant amplitude for gravitational waves. A significant detection of $B-$modes would in fact be a clear signature for inflationary tensor modes and therefore would distinguish the inflationary paradigm over these alternatives.

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